PROBABILITY AND STATISTICS

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AGENDA

- DEFINITION
- THE ALGEBRA OF EVENTS
- AXIOMS OF PROBABILITY
- FURTHER PROPERTIES
- COUNTING OUTCOMES
- PERMUTATIONS
- COMBINATIONS

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SAMPLE SPACES

DEFINITION

The *sample space* is the set of all possible outcomes of an experiment.

EXAMPLE : When we *flip a coin* then sample space is

$$\mathbf{S = \{ H , T \} ,}$$

Where, **H** denotes that the coin lands "Heads up"

and **T** denotes that the coin lands "Tails up".

For a "*fair coin*" we expect H and T to have the same "*chance*" of occurring, i.e., if we flip the coin many times then about 50 % of the outcomes will be **H**.

We say that the *probability* of H to occur is 0.5 (or 50 %) .

The probability of T to occur is then also 0.5.

EXAMPLE

When we *roll a fair die* then the sample space is

$$S = \{1, 2, 3, 4, 5, 6\}.$$

The probability the die lands with k up is $\frac{1}{6}$, (k = 1,2, ,6).

When we roll it 1200 times we expect a 5 up about 200 times.

The probability the die lands with an *even number* up is

$$\frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$$

EXAMPLE

When we toss a coin 3 times and record the results in the *sequence* that they occur, then the sample space is

$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$.

Elements of S are "*vectors*", "*sequences*", or "*ordered outcomes*".

We may expect each of the 8 outcomes to be equally likely.

Thus the probability of the sequence HTT is $\frac{1}{8}$.

The probability of a sequence to contain precisely two Heads are

$$\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{3}{8}.$$

EXAMPLE

When we toss a coin 3 times and record the results without paying attention to the order in which they occur, e.g., if we only record the number of Heads, then the sample space is

$$S = \{H,H,H\},\ \{H,H,T\},\ \{H,T,T\}\ ,\ \{T,T,T\}\ .$$

The outcomes in S are now *sets* ; i.e., order is not important.

Recall that the ordered outcomes are

$$\{\ HHH\ ,\ HHT\ ,\ HTH\ ,\ HTT\ ,\ THH\ ,\ THT\ ,\ TTH\ ,\ TTT\ \}.$$

Note that

$\{H,H,H\}$ corresponds to one of the ordered outcomes,

$\{H,H,T\}$	,,	three	,,
$\{H,T,T\}$	,,	three	,,
$\{T,T,T\}$	,,	one	,,

Thus $\{H,H,H\}$ and $\{T,T,T\}$ each occur with probability $\frac{1}{8}$,

while {H,H,T} and {H,T,T} each occur with probability $\frac{3}{8}$.

EVENTS

In Probability Theory subsets of the sample space are called *events*.

EXAMPLE: The set of basic outcomes of rolling a die once is

$$S = \{1, 2, 3, 4, 5, 6\},$$

so the subset E = { 2 , 4 , 6 } is an example of an event.

If a die is rolled once and it lands with a 2 or a 4 or a 6 up then we say that the event E has *occurred*.

We have already seen that the probability that E occurs is

$$P(E) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}.$$

THE ALGEBRA OF EVENTS

Since events are sets, namely, subsets of the sample space S, we can do the usual set operations :

If E and F are events then we can form

E^c the complement of E

$E \cup F$ the union of E and F

EF the intersection of E and F

We write $E \subset F$ if E is a *subset* of F.

REMARK: In Probability Theory, we use

E^c instead of $\overline{E}$,

EF instead of $E \cap F$,

$E \subset F$ instead of $E \subseteq F$.

SAMPLE SPACES

If the sample space S is *finite* then we typically allow any subset of S to be an event.

EXAMPLE : If we randomly draw *one character* from a box containing the characters *a, b,* and *c,* then the sample space is

$$S = \{a,\ b,\ c\}\ ,$$

and there are 8 possible events, namely, those in the set of events

$$E = \{\},\ \{a\},\ \{b\},\ \{c\}\ ,\ \{a,b\}\ ,\ \{a,c\}\ ,\ \{b,c\}\ ,\ \{a,b,c\}$$

If the outcomes *a, b,* and *c,* are equally likely to occur, then

$$P(\{\ \}) = 0\ , \quad P(\{a\}) = \frac{1}{3}, \quad P(\{b\}) = \frac{1}{3}, \quad P(\{c\}) = \frac{1}{3}\ ,$$

$$P(\{a,b\}) = \frac{2}{3},\ P(\{a,c\}) = \frac{2}{3}\ ,\ P(\{b,c\}) = \frac{2}{3}\ ,\ P(\{a,b,c\}) = 1\ .$$

For example, $P(\{a,b\})$ is the probability the character is an *a* or *a b*

AXIOMS OF PROBABILITY

A *probability function* P assigns a real number (the *probability* of E) to every event E in a sample space S.

P(.) must satisfy the following basic properties :

- $0 \leq P(E) \leq 1$,
- $P(S) = 1$,
- For any *disjoint events* E_i , $i = 1,2, \ldots , n$, we have

$$P(E1 \cup E2 \cup .. \cup En) = P(E1)+P(E2)+ \ldots. +P(En)$$

.

FURTHER PROPERTIES

PROPERTY 1 :

$$P(E \cup Ec) = P(E) + P(Ec) = 1 \text{ . (Why ?)}$$

Thus

$$P(Ec) = 1 - P(E) \text{ .}$$

EXAMPLE:

What is the probability of at least one "H" in *four tosses* of a coin?

SOLUTION:

The sample space S will have16 outcomes. (Which?)

$$P\,(\text{at least one H}) = 1 - P\,(\text{no H}) = 1 - \frac{1}{16} = \frac{15}{16} \text{ .}$$

PROPERTY 2 :

$$P(E \cup F) = P(E) + P(F) - P(EF) .$$

PROOF (using the third axiom) :

$P(E \cup F) = P(EF) + P(EFc) + P(EcF)$

$= [P(EF) + P(EFc)] + [P(EF) + P(EcF)] - P(EF)$

$= P(E) + P(F) - P(EF)$. (Why ?)

NOTE:

- Draw a Venn diagram with E and F to see this !
- The formula is similar to the one for the number of elements :

$$n(E \cup F) = n(E) + n(F) - n(EF) .$$

SAMPLE SPACES

So far our sample spaces S have been *finite.*

S can also be *countably infinite*, e.g., the set Z of all integers.

S can also be *uncountable*, e.g., the set R of all real numbers.

EXAMPLE: Record the low temperature in Montreal on January 8 in each of a large number of years.

We can take S to be the set of *all real numbers*, i.e., S = R.

(Are there are other choices of S ?)

What probability would you expect for the following *events* to have?

$$\text{(a)}\quad P(\{\pi\}) \qquad\qquad \text{(b)}\quad P(\{x : -\pi < x < \pi\})$$

(How does this differ from finite sample spaces?)

We will encounter such infinite sample spaces many times…

COUNTING OUTCOMES

We have seen examples where the outcomes in a *finite* sample space S are *equally likely* , i.e., they have the same probability .

Such sample spaces occur quite often.

Computing probabilities then requires counting all outcomes and counting *certain types* of outcomes .

The counting has to be done carefully!

We will discuss a number of representative examples in detail.

Concepts that arise include *permutations* and *combinations*.

PERMUTATIONS

- Here we count of the number of "*words* " that can be formed from a collection of items (e.g., letters).
- (Also called *sequences , vectors , ordered sets*) .
- The *order* of the items in the word is important; e.g., the word acb is different from the word *bac* .
- The *word length* is the number of characters in the word.

NOTE:

For *sets* the order is not important. For example, the set *{a,c,b}* is the same as the set *{b,a,c}*

EXAMPLE

Suppose that four-letter words of *lower case* alphabetic characters are generated randomly with equally likely outcomes. (Assume that *letters may appear repeatedly*.)

(a) How many four-letter words are there in the sample space S ?

SOLUTION : $26^4 = 456{,}976$.

(b) How many four-letter words are there are there in S that start with the letter ”s ” ?

SOLUTION : 26^3

(c) What is the probability of generating a four-letter word that starts with an ”s ” ?

SOLUTION : $\frac{26^3}{26^4} = \frac{1}{26} \cong 0.038$

Could this have been computed more easily?

EXAMPLE

How many re-orderings (*permutations*) are there of the string *abc* ? (*Here letters may appear only once*.)

SOLUTION : Six, namely, *abc , acb , bac , bca , cab , cba* .

If these permutations are generated randomly with equal probability then what is the probability the word starts with the letter "a " ?

SOLUTION : $$\frac{2}{6} = \frac{1}{3}$$

EXAMPLE: In general, if the word length is n and all characters are distinct then there are n! permutations of the word. (Why ?)

If these permutations are generated randomly with equal probability then what is the probability the word starts with a particular letter ?

SOLUTION : $$\frac{(n-1)!}{n!} = \frac{1}{n}$$. (Why?)

EXAMPLE : How many *words* of length k can be formed from a set of n (distinct) characters, (where $k \leq n$), when letters can be used *at most once* ?

SOLUTION :

$$n\,(n-1)\,(n-2)\,\cdots\,(n-(k-1))$$

$$= n\,(n-1)\,(n-2)\,\cdots\,(n-k+1)$$

$$= \frac{n!}{(n-k)!} \qquad (\text{ Why ? })$$

EXAMPLE : *Three-letter words* are generated randomly from the *five* characters a , b , c , d , e , where letters can be *used at most once.*

(a) How many three-letter words are there in the sample space $\mathcal{S}$?

SOLUTION : $5 \cdot 4 \cdot 3 = 60$.

(b) How many words containing a , b are there in $\mathcal{S}$?

SOLUTION : First place the characters

$$a \ , \ b$$

i.e., *select the two indices* of the locations to place them.

This can be done in

$$3 \times 2 = 6 \text{ ways .} \qquad (\text{ Why ? })$$

There remains one position to be filled with a c , d or an e .

Therefore the number of words is $3 \times 6 = 18$.

(c) Suppose the 60 solutions in the sample space are *equally likely* .

What is the *probability* of generating a three-letter word that contains the letters a and b ?

SOLUTION :

$$\frac{18}{60} = 0.3 .$$

EXERCISE :

Suppose the sample space $\mathcal{S}$ consists of all *five-letter* words having *distinct alphabetic characters* .

- How many words are there in $\mathcal{S}$?
- How many "special" words are in $\mathcal{S}$ for which *only* the second and the fourth character are vowels, *i.e.*, one of $\{a, e, i, o, u, y\}$?
- Assuming the outcomes in $\mathcal{S}$ to be equally likely, what is the probability of drawing such a special word?

COMBINATION

Let S be a set containing n (distinct) elements.

Then

a *combination* of k elements from S,

is

any selection of k elements from S,

where *order is not important*.

(Thus the selection is a *set*.)

NOTE: By definition a *set* always has *distinct elements*.

EXAMPLE :

There are three *combinations* of 2 elements chosen from the set

$$S = \{a , b , c\} ,$$

namely, the *subsets*

$$\{a, b\} , \{a, c\} , \{b, c\} ,$$

whereas there are six *words* of 2 elements from S ,

namely,

$$ab , ba , ac , ca , bc , cb .$$

SAMPLE SPACES

In general, given

a set S of n elements ,

the number of possible subsets of k elements from S equals

$$\binom{n}{k} \equiv \frac{n!}{k!\,(n-k)!} .$$

REMARK : The notation $\binom{n}{k}$ is referred to as

"n *choose* k ".

NOTE : $$\binom{n}{n} = \frac{n!}{n!\,(n-n)!} = \frac{n!}{n!\;0!} = 1 ,$$

since $0! \equiv 1$ (by "convenient definition" !) .

PROOF :

First recall that there are

$$n\ (n-1)\ (n-2)\ \cdots\ (n-k+1)\ \ =\ \ \frac{n!}{(n-k)!}$$

possible *sequences* of k distinct elements from S .

However, every sequence of length k has $k!$ permutations of itself, and each of these defines the same subset of S.

Thus the total number of subsets is

$$\frac{n!}{k!\ (n-k)!}\ \equiv\ \binom{n}{k}\ .$$

EXAMPLE :

In the previous example, with 2 elements chosen from the set

$$\{a \ , \ b \ , \ c\} \ ,$$

we have $n = 3$ and $k = 2$, so that there are

$$\frac{3!}{(3-2)!} \quad = \quad 6 \quad \textit{words} \ ,$$

namely

$$ab \ , \ ba \quad , \quad ac \ , \ ca \quad , \quad bc \ , \ cb \ ,$$

while there are

$$\binom{3}{2} \quad \equiv \quad \frac{3!}{2! \ (3-2)!} \quad = \quad \frac{6}{2} \quad = \quad 3 \quad \textit{subsets} \ ,$$

namely

$$\{a, b\} \quad , \quad \{a, c\} \quad , \quad \{b, c\} \ .$$

EXAMPLE : If we choose 3 elements from $\{a, b, c, d\}$, then

$$n = 4 \text{ and } k = 3,$$

so there are

$$\frac{4!}{(4-3)!} = 24 \quad \text{words, namely :}$$

$$\begin{array}{cccc} abc, & abd, & acd, & bcd, \\ acb, & adb, & adc, & bdc, \\ bac, & bad, & cad, & cbd, \\ bca, & bda, & cda, & cdb, \\ cab, & dab, & dac, & dbc, \\ cba, & dba, & dca, & dcb, \end{array}$$

while there are

$$\binom{4}{3} \equiv \frac{4!}{3!\,(4-3)!} = \frac{24}{6} = 4 \quad \text{subsets},$$

namely,

$$\{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}.$$

EXAMPLE :

(a) How many ways are there to choose a committee of 4 persons from a group of 10 persons, if order is not important?

SOLUTION :

$$\binom{10}{4} = \frac{10!}{4!\,(10-4)!} = 210 \,.$$

(b) If each of these 210 outcomes is equally likely then what is the probability that a particular person is on the committee?

SOLUTION :

$$\binom{9}{3} / \binom{10}{4} = \frac{84}{210} = \frac{4}{10} \,. \qquad (\text{ Why ? })$$

Is this result surprising?

(c) What is the probability that a particular person is *not* on the committee?

SOLUTION :

$$\binom{9}{4} \Big/ \binom{10}{4} = \frac{126}{210} = \frac{6}{10} . \qquad (\text{ Why ? })$$

Is this result surprising?

(d) How many ways are there to choose a committee of 4 persons from a group of 10 persons, if one is to be the chairperson?

SOLUTION :

$$\binom{10}{1} \binom{9}{3} = 10 \binom{9}{3} = 10 \frac{9!}{3!\,(9-3)!} = 840 .$$

QUESTION : Why is this four times the number in (a) ?

EXAMPLE : *Two balls* are selected at random from a bag with *four white* balls and *three black* balls, where order is not important.

What would be an appropriate sample space $\mathcal{S}$?

SOLUTION : Denote the set of balls by

$$B = \{w_1 , w_2 , w_3 , w_4 , b_1 , b_2 , b_3\} ,$$

where same color balls are made "distinct" by numbering them.

Then a good choice of the sample space is

$$\mathcal{S} = \text{the set of } \textit{all subsets} \text{ of } \textit{two balls} \text{ from } B ,$$

because the wording "*selected at random*" suggests that each such subset has the same chance to be selected.

The number of outcomes in $\mathcal{S}$ (which are sets of two balls) is then

$$\binom{7}{2} = 21 .$$

EXAMPLE : (continued $\cdots$)

(*Two balls* are selected at random from a bag with *four white* balls and *three black* balls.)

- What is the probability that both balls are white?

SOLUTION :

$$\binom{4}{2} \Big/ \binom{7}{2} = \frac{6}{21} = \frac{2}{7}.$$

- What is the probability that both balls are black?

SOLUTION :

$$\binom{3}{2} \Big/ \binom{7}{2} = \frac{3}{21} = \frac{1}{7}.$$

- What is the probability that one is white and one is black?

SOLUTION :

$$\binom{4}{1}\binom{3}{1} \Big/ \binom{7}{2} = \frac{4 \cdot 3}{21} = \frac{4}{7}.$$

(Could this have been computed differently?)

EXAMPLE : (continued $\cdots$)

In detail, the sample space $\mathcal{S}$ is

$$
\left\{
\begin{array}{llll|lll}
 & \{w_1,w_2\}, & \{w_1,w_3\}, & \{w_1,w_4\}, & \{w_1,b_1\}, & \{w_1,b_2\}, & \{w_1,b_3\}, \\
 & & \{w_2,w_3\}, & \{w_2,w_4\}, & \{w_2,b_1\}, & \{w_2,b_2\}, & \{w_2,b_3\}, \\
 & & & \{w_3,w_4\}, & \{w_3,b_1\}, & \{w_3,b_2\}, & \{w_3,b_3\}, \\
 & & & & \{w_4,b_1\}, & \{w_4,b_2\}, & \{w_4,b_3\}, \\
 & & & & \text{———} & \text{———} & \text{———} \\
 & & & & & \{b_1,b_2\}, & \{b_1,b_3\}, \\
 & & & & & & \{b_2,b_3\}
\end{array}
\right\}
$$

- $\mathcal{S}$ has 21 outcomes, *each of which is a set* .
- We assumed each outcome of $\mathcal{S}$ has probability $\frac{1}{21}$.
- The *event* "both balls are white" contains 6 outcomes.
- The *event* "both balls are black" contains 3 outcomes.
- The *event* "one is white and one is black" contains 12 outcomes.
- What would be different had we worked with *sequences* ?

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SAMPLE SPACES

EXERCISE :

Three balls are selected at random from a bag containing

2 *red* , 3 *green* , 4 *blue* balls .

What would be an appropriate sample space $\mathcal{S}$?

What is the the number of outcomes in $\mathcal{S}$?

What is the probability that all three balls are *red* ?

What is the probability that all three balls are *green* ?

What is the probability that all three balls are *blue* ?

What is the probability of *one red*, *one green*, and *one blue* ball ?

SAMPLE SPACES

EXAMPLE : A bag contains 4 *black* balls and 4 *white* balls.

Suppose one draws *two balls at the time*, until the bag is empty.

What is the probability that each drawn pair is *of the same color*?

SOLUTION : An *example of an outcome* in the sample space $\mathcal{S}$ is

$$\Big\{ \{w_1, w_3\} , \{w_2, b_3\} , \{w_4, b_1\} , \{b_2, b_4\} \Big\} .$$

The number of such *doubly unordered* outcomes in $\mathcal{S}$ is

$$\frac{1}{4!}\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2} = \frac{1}{4!}\frac{8!}{2!\,6!}\frac{6!}{2!\,4!}\frac{4!}{2!\,2!}\frac{2!}{2!\,0!} = \frac{1}{4!}\frac{8!}{(2!)^4} = 105 \quad \text{(Why?)}$$

The number of such outcomes with *pairwise the same color* is

$$\frac{1}{2!}\binom{4}{2}\binom{2}{2} \cdot \frac{1}{2!}\binom{4}{2}\binom{2}{2} = 3 \cdot 3 = 9 . \qquad (\text{ Why ? })$$

Thus the probability each pair is *of the same color* is $9/105 = 3/35$.

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SAMPLE SPACES

EXAMPLE : (continued $\cdots$)

The 9 outcomes of *pairwise the same color* constitute the *event*

$$\Big\{ \ \{\{w_1,w_2\},\{w_3,w_4\},\{b_1,b_2\},\{b_3,b_4\}\},$$
$$\{\{w_1,w_3\},\{w_2,w_4\},\{b_1,b_2\},\{b_3,b_4\}\},$$
$$\{\{w_1,w_4\},\{w_2,w_3\},\{b_1,b_2\},\{b_3,b_4\}\},$$

$$\{\{w_1,w_2\},\{w_3,w_4\},\{b_1,b_3\},\{b_2,b_4\}\},$$
$$\{\{w_1,w_3\},\{w_2,w_4\},\{b_1,b_3\},\{b_2,b_4\}\},$$
$$\{\{w_1,w_4\},\{w_2,w_3\},\{b_1,b_3\},\{b_2,b_4\}\},$$

$$\{\{w_1,w_2\},\{w_3,w_4\},\{b_1,b_4\},\{b_2,b_3\}\},$$
$$\{\{w_1,w_3\},\{w_2,w_4\},\{b_1,b_4\},\{b_2,b_3\}\},$$
$$\{\{w_1,w_4\},\{w_2,w_3\},\{b_1,b_4\},\{b_2,b_3\}\} \ \Big\} .$$

EXERCISE :

- How many ways are there to choose a committee of 4 persons from a group of 6 persons, if order is not important?
- Write down the list of all these possible committees of 4 persons.
- If each of these outcomes is equally likely then what is the probability that two particular persons are on the committee?

EXERCISE :

Two balls are selected at random from a bag with three white balls and two black balls.

- Show all elements of a suitable sample space.
- What is the probability that both balls are white?

SAMPLE SPACES

EXERCISE :

We are interested in *birthdays* in a class of 60 students.

- What is a good sample space $\mathcal{S}$ for this purpose?
- How many outcomes are there in $\mathcal{S}$?
- What is the probability of *no common birthdays* in this class?
- What is the probability of *common birthdays* in this class?

EXAMPLE :

How many *nonnegative* integer solutions are there to

$$x_1 + x_2 + x_3 = 17 \ ?$$

SOLUTION :

Consider seventeen 1's separated by bars to indicate the possible values of x_1, x_2, and x_3, *e.g.*,

$$111|111111111|11111 \ .$$

The total number of positions in the "display" is $17 + 2 = 19$.

The total number of *nonnegative* solutions is now seen to be

$$\binom{19}{2} = \frac{19!}{(19-2)!\,2!} = \frac{19 \times 18}{2} = 171 \ .$$

EXAMPLE :

How many *nonnegative* integer solutions are there to the *inequality*

$$x_1 + x_2 + x_3 \leq 17 \ ?$$

SOLUTION :

Introduce an *auxiliary variable* (or "*slack variable*")

$$x_4 \equiv 17 - (x_1 + x_2 + x_3) \ .$$

Then

$$x_1 + x_2 + x_3 + x_4 = 17 \ .$$

Use seventeen 1's separated by 3 bars to indicate the possible values of x_1 , x_2 , x_3 , and x_4 , *e.g.*,

$$111|11111111|1111|11 \ .$$

SAMPLE SPACES

$$111|11111111|1111|11 \ .$$

The total number of positions is

$$17 \ + \ 3 \ = \ 20 \ .$$

The total number of *nonnegative* solutions is therefore

$$\binom{20}{3} = \frac{20!}{(20-3)! \, 3!} = \frac{20 \times 19 \times 18}{3 \times 2} = 1140 \ .$$

EXAMPLE :

How many *positive* integer solutions are there to the equation

$$x_1 + x_2 + x_3 = 17 \ ?$$

SOLUTION : Let

$$x_1 = \tilde{x}_1 + 1 \ , \quad x_2 = \tilde{x}_2 + 1 \ , \quad x_3 = \tilde{x}_3 + 1 \ .$$

Then the problem becomes :

How many *nonnegative* integer solutions are there to the equation

$$\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 = 14 \ ?$$

$$111|111111111|11$$

The solution is

$$\binom{16}{2} = \frac{16!}{(16-2)!\ 2!} = \frac{16 \times 15}{2} = 120 \ .$$

EXAMPLE :

What is the probability the *sum* is 9 in *three rolls of a die* ?

SOLUTION : The number of such *sequences* of three rolls with sum 9 is the number of integer solutions of

$$x_1 + x_2 + x_3 = 9 ,$$

with

$$1 \leq x_1 \leq 6 \quad , \quad 1 \leq x_2 \leq 6 \quad , \quad 1 \leq x_3 \leq 6 .$$

Let

$$x_1 = \tilde{x}_1 + 1 \quad , \quad x_2 = \tilde{x}_2 + 1 \quad , \quad x_3 = \tilde{x}_3 + 1 .$$

Then the problem becomes :

How many *nonnegative* integer solutions are there to the equation

$$\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 = 6 ,$$

with

$$0 \leq \tilde{x}_1 , \tilde{x}_2 , \tilde{x}_3 \leq 5 .$$

EXAMPLE : (continued ···)

Now the equation

$$\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 = 6 \ , \qquad (0 \leq \tilde{x}_1 , \tilde{x}_2 , \tilde{x}_3 \leq 5) \ ,$$

$$1|111|11$$

has

$$\binom{8}{2} = 28 \quad \text{solutions} \ ,$$

from which we must *subtract* the 3 *impossible* solutions

$$(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = (6,0,0) \ , \quad (0,6,0) \ , \quad (0,0,6) \ .$$

$$111111|| \ , \quad |111111| \ , \quad ||111111$$

Thus the probability that the *sum* of 3 rolls equals 9 is

$$\frac{28 - 3}{6^3} = \frac{25}{216} \cong 0.116 \ .$$

EXAMPLE : (continued $\cdots$)

The 25 outcomes of the event "*the sum of the rolls is* 9" are

$$\{ \quad 126\,, \quad 135\,, \quad 144\,, \quad 153\,, \quad 162\,,$$
$$216\,, \quad 225\,, \quad 234\,, \quad 243\,, \quad 252\,, \quad 261\,,$$
$$315 \quad 324\,, \quad 333\,, \quad 342\,, \quad 351\,,$$
$$414\,, \quad 423\,, \quad 432\,, \quad 441\,,$$
$$513\,, \quad 522\,, \quad 531\,,$$
$$612\,, \quad 621 \quad \}\,.$$

The "lexicographic" ordering of the *outcomes* (which are *sequences*) in this *event* is used for systematic counting.

EXERCISE :

- How many integer solutions are there to the inequality

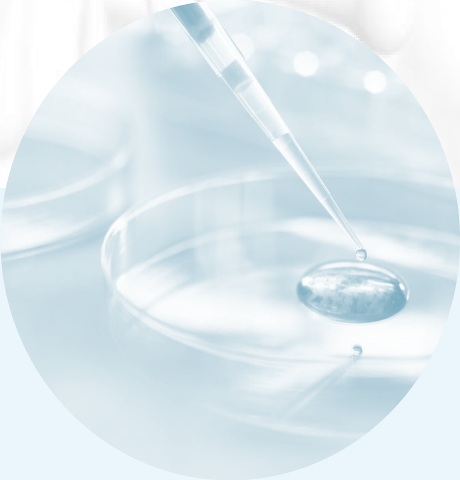
$$x_1 + x_2 + x_3 \leq 17 ,$$

if we require that

$$x_1 \geq 1 , \quad x_2 \geq 2 , \quad x_3 \geq 3 \ ?$$

EXERCISE :

What is the probability that the *sum* is *less than or equal to* 9 in *three rolls of a die* ?



THANK YOU

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