Inventory Optimization

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Agenda

Introduction
- Preliminary
- Objectives

Inventory Models-single period
- Deterministic single period model

Inventory Models - multi-period
- Time-varying Demand
- The Wagner-Whitin Method
- The Silver-Meal Heuristic

Inventory Models-stochastic demand
- Newsboy problem

Multi-echelon Inventories
- Bullwhip Effect

Introduction

According to APICS Dictionary (2015):[1]

Those stocks or items used to support production (raw materials and work-in-process items), supporting activities (maintenance, repair and operating supplies), and customer service (finished goods and spare parts).

Key Milestones:

- 1940s F.W.Harris (Westinghouse) - Lot size formula (EOQ model)
- 1950s Whitin's stochastic extension of the EOQ model
- 1960s Early computers, reorder point (ROP) systems and early material requirements planning (MRP) and manufacturing planning and control (MPC)
- 1970s MRP and computer hardware and software developments- (i) IBM's COPICS (communications oriented production information and control system), (ii) Manufacturing Management and Account System (MMAS) (iii) SAP (Systemanalyse und Programmentwicklung)
- 1990s MRP II and early ERP systems- (i)SAP's R/3 (ii) IBM AS400
- 2000s Software vendor consolidation

Silver, E.A., Pyke, D.F. and Peterson, R., 1998. Inventory management and production planning and scheduling (Vol. 3). New York: Wiley.

[1] *APICS Dictionary. (2015). Fifteenth Edition. Chicago: APICS.*

Introduction

Key challenges:

- How frequently should the inventory status for an item be reviewed?
- When should a replenishment order be placed?
- What should be the order size?

Why inventory planning:

- **Finance:** Keep stocks low to free up investment capital
- **Purchasing:** Order large batches to get volume discounts
- **Production:** Long production runs to avoid time-consuming setups and have a large raw material inventory to avoid production stoppages
- **Marketing:** High stock of finished goods to avoid stockouts

Introduction

Decision models :

- **Static:** Single period problem, classic examples are Christmas tree and newsboy problem
- **Dynamic:** Lot-sizing heuristic
- **Probabilistic:** Newsvendor model

Functions of Inventory:

- **Decoupling Inventory:** Decoupling inventory is an intermediary inventory maintained between two workstations
- **Pipeline Inventory:** Inventory that is in transit (in trucks, ships, etc.) is referred to as pipeline inventory.
- **Cycle Inventory:** Cycle inventory is the result of ordering materials in batches.
- **Buffer Inventory:** Buffer inventory reduces the probability of running out of stock.

Classification:

- **Class A** items should receive the most personalized attention from management. The first 5 to 10% of the SKUs account for 50% or more of the total annual dollar usage ($\sum_i D_i v_i$) of the population of items under consideration
- **Class B** items are of secondary importance in relation to class A. Usually more than 50% of total SKUs account for most of the remaining 50% of the annual dollar usage. monitor using computer-based system with management-by-exception rules
- **Class C** items are the relatively numerous remaining SKUs that make up only a minor part of total dollar investment Decision systems must be kept as simple as possible (eg. two-bin system)

Characterization of items:

- **Demand:** keep stocks low to free up investment capital
- **Replenishment Lead Time:** order large batches to get volume discounts
- **Inventory Level and Review Times:** long production runs to avoid time-consuming setups and have a large raw material inventory to avoid production stoppages
- **Lifetime & Reparability:** have high stock of finished goods to avoid stockouts

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General Inventory Decision Rules

Order frequency	Order quantity	
	Fixed, Q*	Variable, S**
Variable, R***	(Q,R)	(S,R) (min.-max.)
Fixed, T****	(Q,T)	(S,T)

Figure: Types of decision

- **Q:** Order a fixed quantity (Q)
- **S:** Order up to a fixed expected opening inventory quantity (S)
- **R:** Place an order when inventory balance drops to (R)
- **T:** Place an order every (T) periods.

Inventory Models-single period
Deterministic Model

The deterministic models :

- Economic order quantity (EOQ)
- Economic production lot size
- EOQ with quantity discounts
- EOQ with shortages

Assumption for deterministic models :

(1) The demand rate is constant and deterministic.
(2) The order quantity need not be an integral number of units.
(3) The unit variable cost is independent of the replenishment quantity.
(4) The cost factors do not change appreciably with time (i.e. no inflation).
(5) The item is treated entirely independently of other items.
(6) The replenishment lead time is of zero duration.
(7) The entire order quantity is delivered at the same time.

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EOQ models :

The average inventory carrying cost per cycle is the area under the inventory triangle:

$$\frac{1}{2}hQT = \frac{1}{2}h\frac{Q^2}{D}$$

The average cost per cycle is the sum of procurement and inventory carrying cost.

$$A + \frac{1}{2}h\frac{Q^2}{D}$$

To obtain the average annual cost, $TRC(Q)$, we multiply the cost per cycle by the number of cycles per year, D/Q. Doing this and writing $h = vr$, we get

$$TRC(Q) = \frac{AD}{Q} + \frac{rvQ}{2}$$

The optimum value of Q can be found by solving

$$\frac{\partial TRC(Q)}{\partial Q} = -\frac{AD}{Q^2} + \frac{vr}{2} = 0$$

since $TRC(Q)$ is convex function in Q.
(note) A differentiable function $f(x)$ is convex in x, if the second derivative is nonnegative. For the above model,

$$\frac{\partial^2 TRC}{\partial Q^2} = \frac{2AD}{Q^3} \geq 0$$

Consequently,

$$EOQ = \sqrt{\frac{2AD}{vr}}$$

and the minimum average annual cost will be

$$TRC(EOQ) = \sqrt{2ADvr}$$

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Finite Replenishment Rate

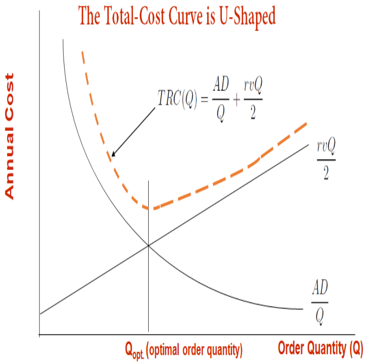

Figure: Cost components

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Inventory Models-single period

Quantity discounts

We assume 'all units discount' which is the most common type of discount structure.

$$v = \begin{cases} v_0 & \text{if } 0 \leq Q < Q_b \\ v_0(1-d) & \text{if } Q_b \leq Q \end{cases}$$

$$TRC(Q) = \frac{Qv_0 r}{2} + \frac{AD}{Q} + Dv_0, \quad 0 \leq Q < Q_b$$

$$TRC(Q) = \frac{Qv_0(1-d)r}{2} + \frac{AD}{Q} + Dv_0(1-d), \quad Q \geq Q_b$$

♣ tradeoff between extra carrying cost vs. a reduction in the acquisition costs

♣ Solution steps

- Step 1

  $$\text{Compute} \quad \text{EOQ(discount)} = \sqrt{\frac{2AD}{v_0(1-d)r}}$$

- Step 2 If $EOQ(d) \geq Q_b$, then EOQ(d) is optimal (case (c)). If $EOQ(d) < Q_b$, go to Step 3.
- Step 3 Compute TRC(EOQ) and TRC(Q_b).
  If TRC(EOQ) $\leq$ TRC(Q_b), EOQ is optimal (case (b)).
  If TRC(EOQ) $>$ TRC(Q_b), Q_b is optimal (case (a)).

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Two products

Let us assume, T is the same for both items, we get: $T = \frac{Q_1}{D_1} = \frac{Q_2}{D_2}$.

The average inventory carrying cost for item 1 per cycle is : $\frac{1}{2} r v_1 Q_1 T = \frac{1}{2} r v_1 \frac{Q_1^2}{D_1}$

The average inventory carrying cost for item 2 per cycle is : $\frac{1}{2} r v_2 Q_2 T = \frac{1}{2} r v_2 \frac{Q_2^2}{D_2}$

The average cost per cycle is the sum of procurement and inventory carrying cost: $A + \frac{1}{2} r v_1 \frac{Q_1^2}{D_1} + \frac{1}{2} r v_2 \frac{Q_2^2}{D_2}$

To obtain the average annual cost, $TRC(Q_1, Q_2)$, we multiply the cost per cycle by the number of cycles per year, $D_1/Q_1 (\equiv D_2/Q_2)$. Doing this, we get : $TRC(Q_1, Q_2) = \frac{AD_1}{Q_1} + \frac{r v_1 Q_1}{2} + \frac{r v_2 Q_2}{2}$

By substituting $Q_2 = \frac{D_2}{D_1} Q_1$, we get

$$TRC(Q_1) = \frac{AD_1}{Q_1} + \frac{r v_1 Q_1}{2} + \frac{r v_2 D_2 Q_1}{2 D_1}$$

The optimum value of Q_1 can be found by solving

$$\frac{\partial TRC}{\partial Q_1} = -\frac{AD_1}{Q_1^2} + \frac{r(v_1 D_1 + c_2 D_2)}{2 D_1} = 0$$

$TRC(Q_1)$ is a convex function in Q_1, $\frac{\partial^2 TRC}{\partial Q_1^2} = \frac{2AD_1}{Q_1^3} \geq 0$, consequently,

$$Q_1^* = \sqrt{\frac{2AD_1^2}{r(v_1 D_1 + v_2 D_2)}} \quad Q_2^* = \sqrt{\frac{2AD_2^2}{r(v_1 D_1 + v_2 D_2)}} \quad T^* = \sqrt{\frac{2A}{r(v_1 D_1 + v_2 D_2)}}$$

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Inventory Models-single period

Exercises

A company has a chance to reduce their inventory ordering costs by placing larger quantity orders using the price-break order quantity schedule below. What should their optimal order quantity be if this company purchases this single inventory item with an e-mail ordering cost of $4, a carrying cost rate of 2% of the inventory cost of the item, and an annual demand of 10,000 units?

Order Quantity(units)	Price/unit
0 to 2,499	1.20
2,500 to 3,999	1.00
4,000 or more	0.98

Inventory Models - multi-period
Time-varying Demand

In the basic inventory models, deterministic and level demand rates are assumed. Here we allow the average demand rate to vary with time, thus encompassing a broader range of practical situations such as :

- Multi-echelon assembly operations where a firm schedule of finished products exploded back through the various assembly stages leads to production requirements at these earlier levels, which are relatively deterministic but almost always vary appreciably with time.
- Production to contract, where the contract requires that certain quantities have to be delivered to the consumer on specified dates.

We will learn the following Approaches:

- Lot-for-Lot policy
- straight-forward use of the economic order quantity
- an exact optimal procedure (Wagner-Whitin Algorithm
- an approximate heuristic method (Silver-Meal Heuristic)

Inventory Models - multi-period

Consider the selection of the replenishment quantity at the beginning of January. See the following table.

Month	January	February	March	April	May	June
Demand	10	62	12	130	26	80
Cumulative Demand	10	72	84	214	240	320

You may have a question that why we need to seek a heuristic algorithm even though we have an exact optimal algorithm to solve the problem. The answer is owing to the complexity of the optimal algorithm. The complexity of the exact optimal algorithm is exponential. That is, we cannot solve large problems using the algorithm. Also, it requires an additional assumption which will be explained later.

Inventory Models - multi-period

General assumptions:

- The demand rate is given in the form of D_j to be satisfied in period j ($j = 1, \ldots, N$) where the planning horizon is at the end of period N. Of course, the demand rate may vary from one period to the next, but it is assumed known.
- The entire requirements of each period must be available at the beginning of that period.
- The unit variable cost does not depend on the replenishment quantity.
- Inflation is at a negligibly low level
- The item is treated entirely independently of other items
- The replenishment lead time is known with certainty
- No shortages are allowed
- The carrying cost is only applicable to inventory that is carried over from one period to the next.

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The Wagner-Whitin Method :
Wagner and Whitin (1958) developed an algorithm that guarantees an optimal solution of replenishment quantities under one additional assumption : Either the demand pattern terminates at the horizon or else the ending inventory must be prespecified.

The algorithm is an application of dynamic programming, a mathematical procedure for solving sequential decision problems. Suppose we define $F(t)$ as the total cost of the best replenishment strategy that satisfies the demand requirements in periods $1, 2, \ldots, t$. To illustrate the procedure for finding $F(t)$, we again use the example.

$F(1)$ is the total cost of a replenishment of size 10 at the start of January, simply the setup cost A or \$54.
We continue forward in this fashion until we complete period N. For any specific month t there are t possible options to evaluate. Note that the method requires an ending point where it is known that the inventory level is to be at zero or some other specified value.

Mathematically, we can represent the preceding procedures as follows :

$$F(t) = \text{Min}_{0 \leq j \leq t-1}[F(j) + c_{jt}]$$

where c_{jt} = cost in period $j+1$ to satisfy demands in period $j+1, \ldots, t$.

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The Silver-Meal or Least Period Cost Heuristic:
The Wagner-Whitin algorithm has some drawbacks from the practitioner's standpoint. Therefore, the natural question to ask is "Is there a simpler approach that will capture most of the potential savings?" Silver and Meal (1976) have developed a simple variation of the basic EOQ which accomplishes exactly what we desire. Moreover, in numerous test examples the Silver-Meal heuristic has performed extremely well when compared with the other rules encountered in the literature.
Let the total relevant costs associated with a replenishment that lasts for T periods be denoted by TRC(T). We wish to select T to minimize the total relevant costs per unit time, TRCUT(T), where

$$\text{TRCUT}(T) = \frac{\text{TRC}(T)}{T} = \frac{A + \text{ carrying costs}}{T}$$

The basic idea of the heuristic is to evaluate TRCUT(T) for increasing values of T until, for the first time,

$$\text{TRCUT}(T+1) > \text{TRCUT}(T)$$

that is, the total relevant costs per unit time start increasing. When this happens the associated T is selected as the number of periods that the replenishment should cover.

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Play with your Friends

Lets have a Fun !!!!!

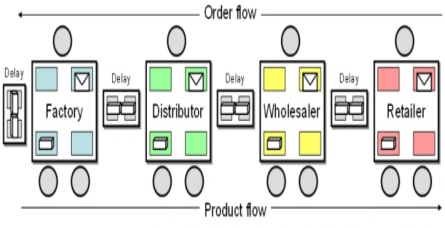

Figure: Multi-echelon system

Inventory Models-stochastic demand
Newsboy problem

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Main features of the style goods problem

- There is a relatively short selling season with a well-defined beginning and end.
- Buyers or producers have to commit themselves the order quantity or production quantity prior to the start of the selling season.
- Forecasts prior to the season include considerable uncertainty stemming from the long period of inactivity (no sales) between seasons.
- When the total demand in the season exceeds the stock made available, there are associated underage costs.

♣ Notation:

- v= acquisition cost, in dollars/unit
- p= selling price, in dollars/unit
- B (or $B_2 v$)= penalty for not satisfying demand, in dollars/unit
- g= salvage value, in dollars/unit
- X= demand (random variable)
- $\hat{x}$= expected demand
- σ_X= standard deviation of demand
- $p_{X<}(x_0)$= cumulative distribution of total demand
- $X^+ = \max\{X, 0\}$, $\min(Q, X) = X - (X - Q)^+$, $(Q - X)^+ = (Q - X) + (X - Q)^+$,
- Q=quantity to be stocked, in units (decision variable)

Inventory Models-stochastic demand

we can write the expected profits as

$$\begin{aligned}\pi^F(Q) &= p\hat{x} - pE\left(X-Q\right)^+ + gQ - g\hat{x} + gE\left(X-Q\right)^+ - BE\left(X-Q\right)^+ - vQ \\ &= (p-g)\hat{x} - (v-g)Q - (p-g+B)E\left(X-Q\right)^+\end{aligned}$$

By applying Leibnitz's rule, we get the following result:

$$\frac{d\pi^F(Q)}{dQ} = -(v-g)-(p-g+B)[-p_{x>}(Q)] = -(v-g)-(p-g+B)[p_{x<}(Q)-1] = 0$$

$$p_{x<}(Q^*) = \frac{p-v+B}{p-g+B}$$

The Case of Normally Distributed Demand

$$k = \frac{Q-\hat{x}}{\sigma_x}$$

$p_{u\geq}(k)$= probability that a unit normal variable takes on a value of k or larger

$$p_{u<}(k) = \frac{p-v+B}{p-g+B} \longrightarrow 1 - p_{u\geq}(k) = \frac{p-v+B}{p-g+B} \longrightarrow p_{u\geq}(k) = \frac{v-g}{p-g+B}$$

$$Q = \hat{x} + k\sigma_x$$

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Multi-echelon Inventories

Where inventory should be....??

- Inventory variability increases in moving up the supply chain from consumer to grocery store to distribution center to central warehouse to factory
- One of the main causes is that retailers and distributors often overact to shortages by ordering more than they need.
- (Example) In the Italian pasta industry, demand is quite steady throughout the year. However, because of trade promotions, volume discounts, long lead times, full-truckload discounts, and end-of-quarter sales incentives the orders seen at the manufacturers are highly variable.

Sequential Stocking Points with Level Demand
● the simplest of multi-echelon situations where the stocking points are serially connected. (eg.) one central warehouse, one retailer warehouse, and one retail outlet.

Multi-echelon Inventories

Echelon stock of echelon j: the number of units in the system that are at, or have passed through, echelon j but have as yet not been specifically committed to outside customers $\rightarrow$ each echelon stock has a sawtooth pattern with time $\rightarrow$ the same physical units of stock can appear in more than one echelon inventory $\rightarrow$ value any specific echelon inventory at only the value added at that particular echelon

$$v'_W = v_W \quad v'_R = v_R - v_W$$

$$v'_i = v_i - \sum_{j \in P} v_j$$

where P ={all immediate predecessors of i}

$$TRC(Q_W, Q_R) = \frac{A_W D}{Q_W} + \bar{I}'_W v'_W r + \frac{A_R D}{Q_R} + \bar{I}'_R v'_R r$$

where
$\bar{I}'_W$ = average value of the warehouse echelon inventory, in units
$\bar{I}'_R$ = average value of the retailer echelon inventory, in units

$$TRC(n, Q_R) = \frac{D}{Q_R}\left(A_R + \frac{A_W}{n}\right) + \frac{Q_R r}{2}\left(n v'_W + v'_R\right)$$

$$Q^*_R(n) = \sqrt{\frac{2\left(A_R + \frac{A_W}{n}\right) D}{(n v'_W + v'_R) r}}$$

Multi-echelon Inventories

Minimize $TRC^*(n) \equiv$ Minimize $F(n)$ where

$$F(n) = \left[A_R + \frac{A_W}{n}\right](nv'_W + v'_R)$$

$$\frac{dF(n)}{dn} = (nv'_W + v'_R)\left(-\frac{A_W}{n^2}\right) + \left[A_R + \frac{A_W}{n}\right]v'_W = 0 \longrightarrow n^* = \sqrt{\frac{A_W v'_R}{A_R v'_W}}$$

(Step 1) Compute

$$n^* = \sqrt{\frac{A_W v'_R}{A_R v'_W}}$$

If n^* is exactly an integer, go to (Step 4) with $n = n^*$. Also, if $n^* < 1$, go to (Step 4) with n=1. Otherwise, proceed to (Step 2).

(Step 2) Ascertain the two integer values, n_1 and n_2 that surround n^*.

(Step 3) Evaluate

$$F(n_1) = \left[A_R + \frac{A_W}{n_1}\right](n_1 v'_W + v'_R)$$

$$F(n_2) = \left[A_R + \frac{A_W}{n_2}\right](n_2 v'_W + v'_R)$$

If $F(n_1) \leq F(n_2)$, use $n = n_1$.
If $F(n_1) > F(n_2)$, use $n = n_2$.
(Step 4) Evaluate

$$Q_R = \sqrt{\frac{2\left(A_R + \frac{A_W}{n}\right)D}{(nv'_W + v'_R)r}}$$

(Step 5) Calculate

$$Q_W = nQ_R$$

Discuss possible solution approaches

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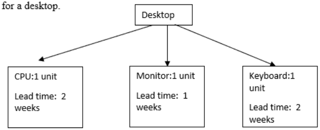
Problem 1

Consider a computer manufacturing firm that uses a Materials Requirement Planning (MRP) system for desktops. To illustrate how the two optimization methods (e.g., EOQ, Wagner–Whitin, or you can use another) could be incorporated into an MRP system, consider the following simple product structure for a desktop.



Assume that we are in week 1 of the current planning horizon. Suppose the projected net requirement of the desktops (weeks 3 through 9) are as follows:

Week	3	4	5	6	7	8	9
Demand	200	150	140	210	200	100	190

Using the product structure, the *planned order release* (that is, the requirement) for CPU units (given a two-week lead time) can be generated as follows:

Week	1	2	3	4	5	6	7
Demand	200	150	140	210	200	100	190

Determine a production schedule for CPU units using the Wagner–Whitin algorithm. Take the holding cost to be $1 per unit per period and the fixed cost of production to be $250. Show also the impact of ignoring the lead time effect. Can you compute safety stock also?
You can use unit cost information for components from any source if required.

Thanks

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