PROBABILITY AND STATISTICS

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AGENDA

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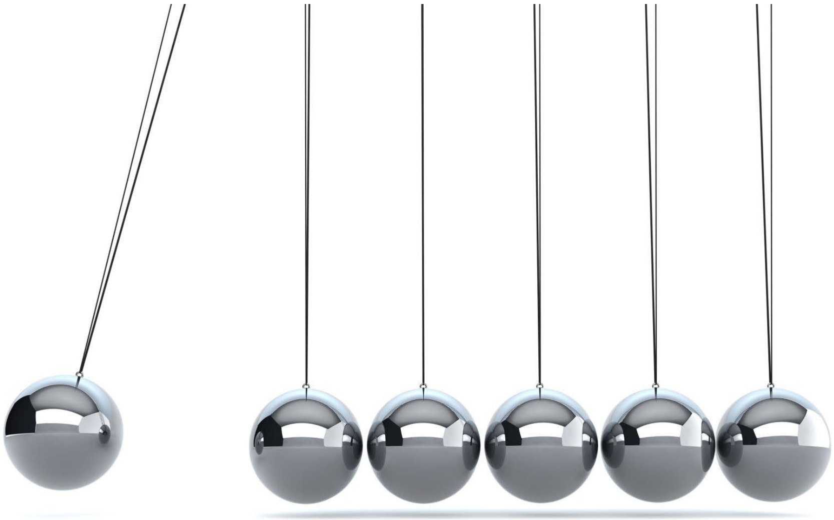

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CONTINUOUS RANDOM VARIABLES

DEFINITION : A *continuous random variable* is a *function* $X(s)$ from an *uncountably infinite* sample space $\mathcal{S}$ to the real numbers $\mathbb{R}$,

$$X(\cdot) \quad : \quad \mathcal{S} \quad \to \quad \mathbb{R} \; .$$

EXAMPLE :

Rotate a *pointer* about a pivot in a plane (like a hand of a clock).

The *outcome* is the *angle* where it stops : $2\pi\theta$, where $\theta \in (0,1]$.

A good *sample space* is all values of θ , *i.e.* $\mathcal{S} = (0,1]$.

A very simple example of a *continuous random variable* is $X(\theta) = \theta$.

Suppose *any outcome*, *i.e.*, any value of θ is "equally likely".

What are the values of

$$P(0 < \theta \leq \tfrac{1}{2}) \quad , \quad P(\tfrac{1}{3} < \theta \leq \tfrac{1}{2}) \quad , \quad P(\theta = \tfrac{1}{\sqrt{2}}) \; ?$$

The (*cumulative*) *probability distribution function* is defined as

$$F_X(x) \equiv P(X \leq x) .$$

Thus

$$F_X(b) - F_X(a) \equiv P(a < X \leq b) .$$

We must have

$$F_X(-\infty) = 0 \quad \text{and} \quad F_X(\infty) = 1 ,$$

i.e.,

$$\lim_{x \to -\infty} F_X(x) = 0 ,$$

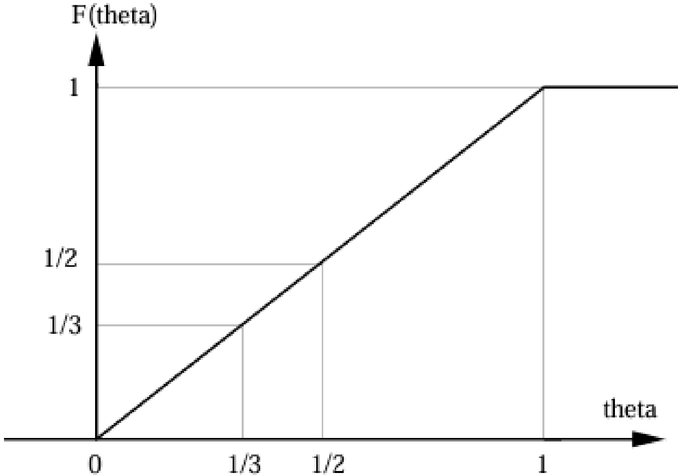
and

$$\lim_{x \to \infty} F_X(x) = 1 .$$

Also, $F_X(x)$ is a *non-decreasing* function of x . (Why ?)

NOTE : All the above is *the same* as for *discrete* random variables !

EXAMPLE : In the "*pointer example*", where $X(\theta) = \theta$, we have the *probability distribution function*



Note that

$$F(\tfrac{1}{3}) \equiv P(X \leq \tfrac{1}{3}) = \tfrac{1}{3} \quad , \quad F(\tfrac{1}{2}) \equiv P(X \leq \tfrac{1}{2}) = \tfrac{1}{2} \, ,$$

$$P(\tfrac{1}{3} < X \leq \tfrac{1}{2}) = F(\tfrac{1}{2}) - F(\tfrac{1}{3}) = \tfrac{1}{2} - \tfrac{1}{3} = \tfrac{1}{6} \, .$$

QUESTION : What is $P(\frac{1}{3} \leq X \leq \frac{1}{2})$?

The *probability density function* is the *derivative* of the probability distribution function :

$$f_X(x) \equiv F'_X(x) \equiv \frac{d}{dx} F_X(x) .$$

EXAMPLE : In the "*pointer example*"

$$F_X(x) = \begin{cases} 0 , & x \leq 0 \\ x , & 0 < x \leq 1 \\ 1 , & 1 < x \end{cases}$$

Thus

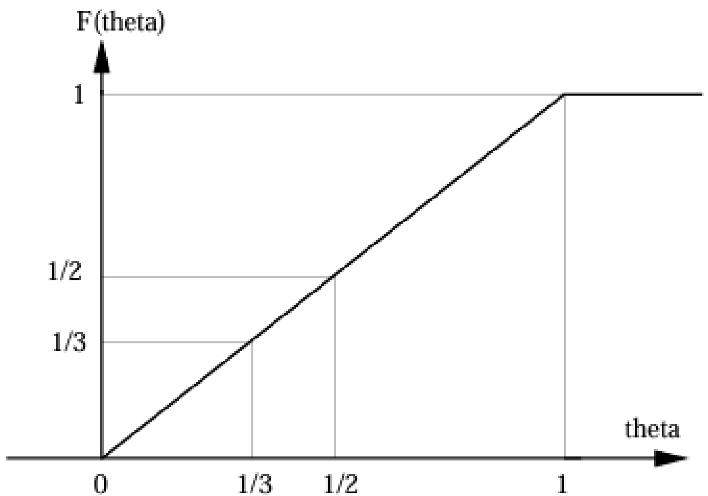
$$f_X(x) = F'_X(x) = \begin{cases} 0 , & x \leq 0 \\ 1 , & 0 < x \leq 1 \\ 0 , & 1 < x \end{cases}$$

NOTATION : When it is clear what X is then we also write

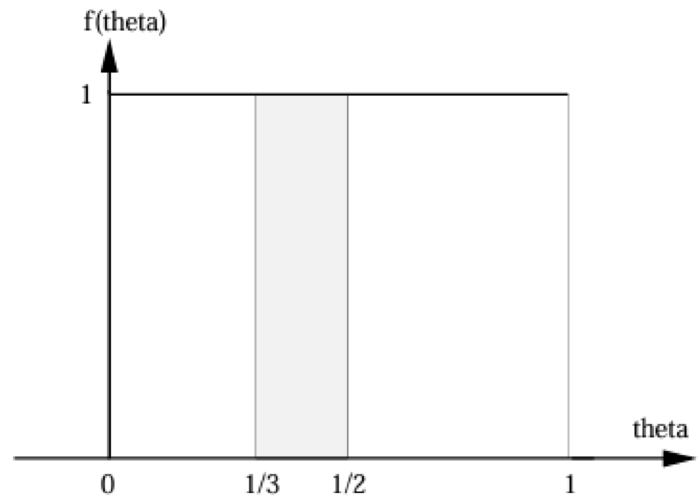
$$f(x) \quad \text{for} \quad f_X(x) , \qquad \text{and} \qquad F(x) \quad \text{for} \quad F_X(x) .$$

EXAMPLE : (continued ···)

$$F(x) = \begin{cases} 0\,, & x \leq 0 \\ x\,, & 0 < x \leq 1 \\ 1\,, & 1 < x \end{cases} \quad , \quad f(x) = \begin{cases} 0\,, & x \leq 0 \\ 1\,, & 0 < x \leq 1 \\ 0\,, & 1 < x \end{cases}$$



Distribution function



Density function

NOTE :

$$P\left(\frac{1}{3} < X \leq \frac{1}{2}\right) = \int_{\frac{1}{3}}^{\frac{1}{2}} f(x)\, dx = \frac{1}{6} = \text{the shaded area .}$$

In general, from

$$f(x) \equiv F'(x) ,$$

with

$$F(-\infty) = 0 \quad \text{and} \quad F(\infty) = 1 ,$$

we have from Calculus the following *basic identities* :

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{\infty} F'(x)\, dx = F(\infty) - F(-\infty) = 1 ,$$

$$\int_{-\infty}^{x} f(x)\, dx = F(x) - F(-\infty) = F(x) = P(X \le x) ,$$

$$\int_{a}^{b} f(x)\, dx = F(b) - F(a) = P(a < X \le b) ,$$

$$\int_{a}^{a} f(x)\, dx = F(a) - F(a) = 0 = P(X = a) .$$

EXERCISE : Draw *graphs* of the distribution and density functions

$$F(x) = \begin{cases} 0 \,, & x \le 0 \\ 1 - e^{-x} \,, & x > 0 \end{cases} \quad , \quad f(x) = \begin{cases} 0 \,, & x \le 0 \\ e^{-x} \,, & x > 0 \end{cases} \quad ,$$

and verify that

- $F(-\infty) = 0 \,, \qquad F(\infty) = 1 \,,$
- $f(x) = F'(x) \,,$
- $F(x) = \int_0^x f(x)\,dx \,,$ (Why is *zero* as lower limit OK ?)
- $\int_0^\infty f(x)\,dx = 1 \,,$
- $P(0 < X \le 1) = F(1) - F(0) = F(1) = 1 - e^{-1} \cong 0.63 \,,$
- $P(X > 1) = 1 - F(1) = e^{-1} \cong 0.37 \,,$
- $P(1 < X \le 2) = F(2) - F(1) = e^{-1} - e^{-2} \cong 0.23 \,.$

EXERCISE : For positive integer n, consider the density functions

$$f_n(x) = \begin{cases} cx^n(1-x^n)\ , & 0 \le x \le 1 \\ 0\ , & \text{otherwise} \end{cases}$$

- Determine the value of c in terms of n .
- Draw the graph of $f_n(x)$ for $n = 1, 2, 4, 8, 16$.
- Determine the distribution function $F_n(x)$.
- Draw the graph of $F_n(x)$ for $n = 1, 2, 3, 4, 8, 16$.
- Determine $P(0 \le X \le \frac{1}{2})$ in terms of n .
- What happens to $P(0 \le X \le \frac{1}{2})$ when n becomes large?
- Determine $P(\frac{9}{10} \le X \le 1)$ in terms of n .
- What happens to $P(\frac{9}{10} \le X \le 1)$ when n becomes large?

Joint distributions

A *joint probability density function* $f_{X,Y}(x,y)$ must satisfy

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy \;=\; 1 \qquad (\text{"Volume"} = 1\ ).$$

The corresponding *joint probability distribution function* is

$$F_{X,Y}(x,y) \;=\; P(X \le x\ ,\ Y \le y) \;=\; \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(x,y)\,dx\,dy\ .$$

By Calculus we have $\dfrac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y} \;=\; f_{X,Y}(x,y)\ .$

Also,

$$P(a < X \le b\ ,\ c < Y \le d) \;=\; \int_{c}^{d}\int_{a}^{b} f_{X,Y}(x,y)\,dx\,dy\ .$$

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EXAMPLE :

If

$$f_{X,Y}(x,y) = \begin{cases} 1 & \text{for } x \in (0,1] \text{ and } y \in (0,1] \,, \\ 0 & \text{otherwise} \,, \end{cases}$$

then, for $x \in (0,1]$ and $y \in (0,1]$,

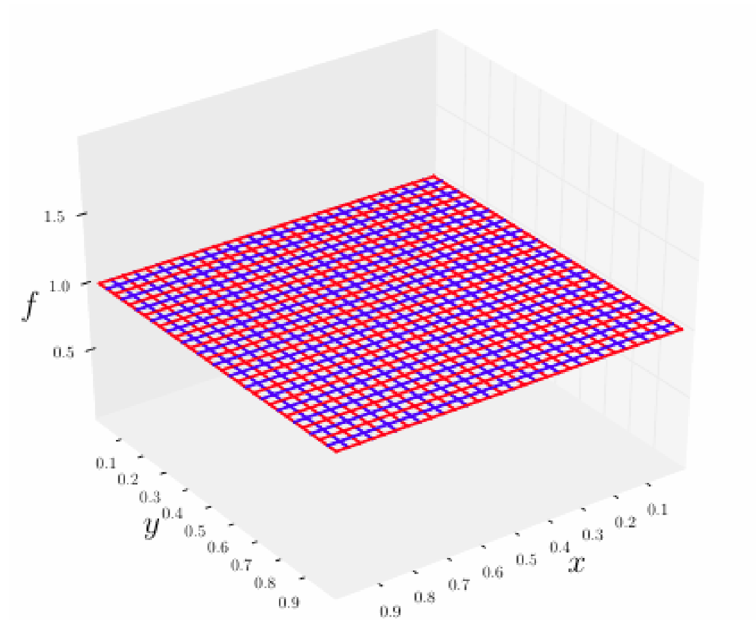
$$F_{X,Y}(x,y) = P(X \le x \,,\, Y \le y) = \int_0^y \int_0^x 1 \, dx \, dy = xy \,.$$

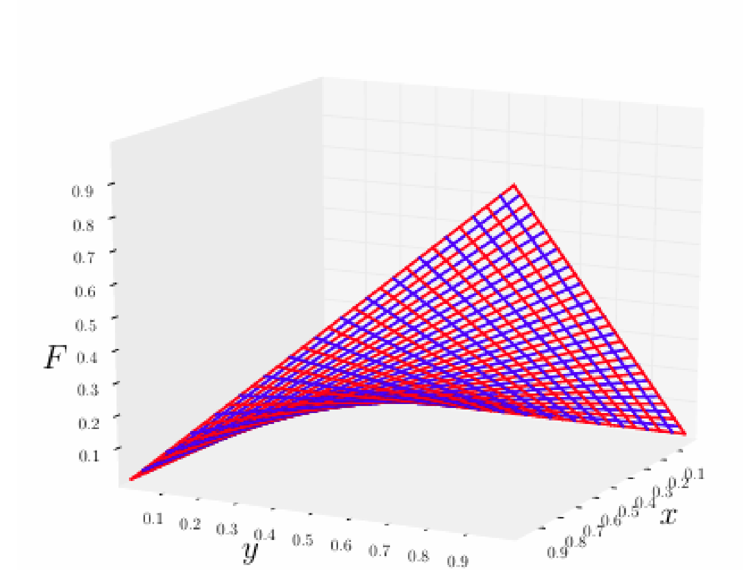
Thus

$$F_{X,Y}(x,y) = xy \,, \qquad \text{for } x \in (0,1] \text{ and } y \in (0,1] \,.$$

For example

$$P(\, X \le \frac{1}{3} \,,\, Y \le \frac{1}{2} \,) = F_{X,Y}(\, \frac{1}{3} \,,\, \frac{1}{2} \,) = \frac{1}{6} \,.$$





Also,

$$P\left(\frac{1}{3} \leq X \leq \frac{1}{2}\,,\ \frac{1}{4} \leq Y \leq \frac{3}{4}\right) \;=\; \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{3}}^{\frac{1}{2}} f(x,y)\,dx\,dy \;=\; \frac{1}{12}\,.$$

EXERCISE : Show that we can also compute this as follows :

$$F(\tfrac{1}{2},\tfrac{3}{4}) \;-\; F(\tfrac{1}{3},\tfrac{3}{4}) \;-\; F(\tfrac{1}{2},\tfrac{1}{4}) \;+\; F(\tfrac{1}{3},\tfrac{1}{4}) \;=\; \tfrac{1}{12}\,.$$

and explain why !

Marginal density functions

The *marginal density functions* are

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy \quad , \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\ .$$

with corresponding *marginal distribution functions*

$$F_X(x) \equiv P(X \leq x) = \int_{-\infty}^{x} f_X(x)\,dx = \int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy\,dx\ ,$$

$$F_Y(y) \equiv P(Y \leq y) = \int_{-\infty}^{y} f_Y(y)\,dy = \int_{-\infty}^{y}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy\ .$$

By Calculus we have

$$\frac{dF_X(x)}{dx} = f_X(x) \quad , \quad \frac{dF_Y(y)}{dy} = f_Y(y)\ .$$

EXAMPLE : If

$$f_{X,Y}(x,y) = \begin{cases} 1 & \text{for } x \in (0,1] \text{ and } y \in (0,1] \,, \\ 0 & \text{otherwise} \,, \end{cases}$$

then, for $x \in (0,1]$ and $y \in (0,1]$,

$$f_X(x) = \int_0^1 f_{X,Y}(x,y)\,dy = \int_0^1 1\,dy = 1 \,,$$

$$f_Y(y) = \int_0^1 f_{X,Y}(x,y)\,dx = \int_0^1 1\,dx = 1 \,,$$

$$F_X(x) = P(X \le x) = \int_0^x f_X(x)\,dx = x \,,$$

$$F_Y(y) = P(Y \le y) = \int_0^y f_Y(y)\,dy = y \,.$$

For example

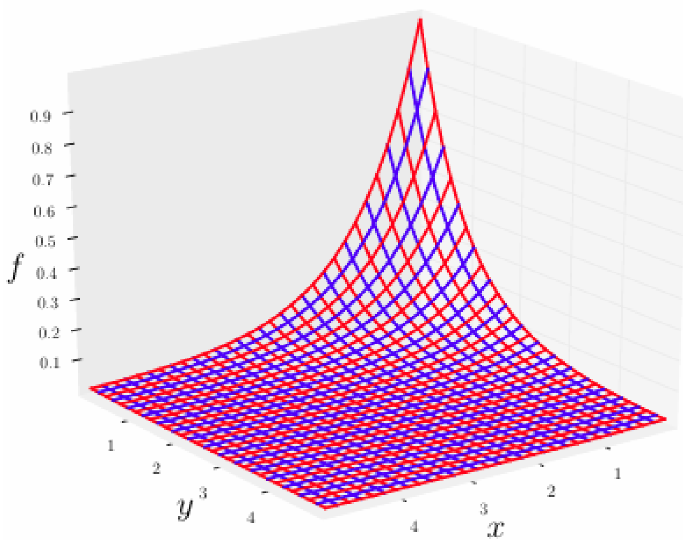
$$P(\,X \le \tfrac{1}{3}\,) = F_X(\tfrac{1}{3}) = \tfrac{1}{3} \quad , \quad P(\,Y \le \tfrac{1}{2}\,) = F_Y(\tfrac{1}{2}) = \tfrac{1}{2} \,.$$

EXERCISE :

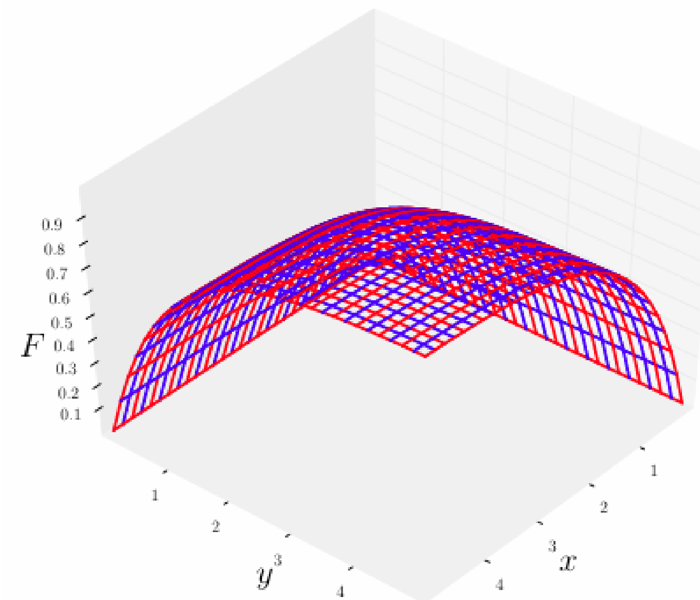
Let $$F_{X,Y}(x,y) = \begin{cases} (1-e^{-x})(1-e^{-y}) & \text{for } x \geq 0 \text{ and } y \geq 0 , \\ 0 & \text{otherwise} . \end{cases}$$

- Verify that

$$f_{X,Y}(x,y) = \frac{\partial^2 F}{\partial x \partial y} = \begin{cases} e^{-x-y} & \text{for } x \geq 0 \text{ and } y \geq 0 , \\ 0 & \text{otherwise} . \end{cases}$$



Density function $f_{X,Y}(x,y)$



Distribution function $F_{X,Y}(x,y)$

EXERCISE : (continued $\cdots$)

$$F_{X,Y}(x,y) = (1-e^{-x})(1-e^{-y}) \quad , \quad f_{X,Y}(x,y) = e^{-x-y} \, , \quad \text{for } x, y \geq 0 \, .$$

Also verify the following :

- $F(0,0) = 0 \quad , \quad F(\infty,\infty) = 1$,
- $\int_0^\infty \int_0^\infty f_{X,Y}(x,y)\, dx\, dy = 1$, (Why *zero* lower limits ?)
- $f_X(x) = \int_0^\infty e^{-x-y}\, dy = e^{-x}$,
- $f_Y(y) = \int_0^\infty e^{-x-y}\, dx = e^{-y}$.
- $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$. (So ?)

EXERCISE : (continued ⋯)

$$F_{X,Y}(x,y) = (1-e^{-x})(1-e^{-y}) \quad , \quad f_{X,Y}(x,y) = e^{-x-y} \, , \quad \text{for } x, y \geq 0 \, .$$

Also verify the following :

- $F_X(x) = \int_0^x f_X(x)\, dx = \int_0^x e^{-x}\, dx = 1 - e^{-x}$,
- $F_Y(y) = \int_0^y f_Y(y)\, dy = \int_0^y e^{-y}\, dy = 1 - e^{-y}$,
- $F_{X,Y}(x,y) = F_X(x) \cdot F_Y(y)$. (So ?)
- $P(1 < x < \infty) = F_X(\infty) - F_X(1) = 1 - (1 - e^{-1}) = e^{-1} \cong 0.37$,
- $P(\, 1 < x \leq 2 \, , \; 0 < y \leq 1 \,) = \int_0^1 \int_1^2 e^{-x-y}\, dx\, dy$

$$= (e^{-1} - e^{-2})(1 - e^{-1}) \cong 0.15 \, ,$$

Independent continuous random variables

Recall that two events E and F are *independent* if

$$P(EF) \quad = \quad P(E)\, P(F)\ .$$

Continuous random variables $X(s)$ and $Y(s)$ are *independent* if

$$P(X \in I_X\ ,\ Y \in I_Y) \quad = \quad P(X \in I_X) \cdot P(Y \in I_Y)\ ,$$

for *all* allowable sets I_X and I_Y (typically *intervals*) of *real numbers*.

Equivalently, $X(s)$ and $Y(s)$ are independent if for all such sets I_X and I_Y the *events*

$$X^{-1}(I_X) \quad \text{and} \quad Y^{-1}(I_Y)\ ,$$

are independent *in the sample space* $\mathcal{S}$.

NOTE :

$$X^{-1}(I_X) \equiv \{s \in \mathcal{S}\ :\ X(s) \in I_X\}\ ,$$
$$Y^{-1}(I_Y) \equiv \{s \in \mathcal{S}\ :\ Y(s) \in I_Y\}\ .$$

FACT : $X(s)$ and $Y(s)$ are *independent* if for all x and y

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y) .$$

EXAMPLE : The random variables with density function

$$f_{X,Y}(x,y) = \begin{cases} e^{-x-y} & \text{for } x \geq 0 \text{ and } y \geq 0 , \\ 0 & \text{otherwise} , \end{cases}$$

are *independent* because (by the preceding exercise)

$$f_{X,Y}(x,y) = e^{-x-y} = e^{-x} \cdot e^{-y} = f_X(x) \cdot f_Y(y) .$$

NOTE :

$$F_{X,Y}(x,y) = \begin{cases} (1-e^{-x})(1-e^{-y}) & \text{for } x \geq 0 \text{ and } y \geq 0 , \\ 0 & \text{otherwise} , \end{cases}$$

also satisfies (by the preceding exercise)

$$F_{X,Y}(x,y) = F_X(x) \cdot F_Y(y) .$$

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PROPERTY :

For *independent* continuous random variables X and Y we have

$$F_{X,Y}(x,y) = F_X(x) \cdot F_Y(y) , \quad \text{for all } x, y .$$

PROOF :

$$\begin{aligned}
F_{X,Y}(x,y) &= P(X \le x , \ Y \le y) \\
&= \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(x,y) \, dy \, dx \\
&= \int_{-\infty}^{x} \int_{-\infty}^{y} f_X(x) \cdot f_Y(y) \, dy \, dx \qquad \text{(by independence)} \\
&= \int_{-\infty}^{x} [\, f_X(x) \cdot \int_{-\infty}^{y} f_Y(y) \, dy \,] \, dx \\
&= [\int_{-\infty}^{x} f_X(x) \, dx \,] \cdot [\int_{-\infty}^{y} f_Y(y) \, dy \,] \\
&= F_X(x) \cdot F_Y(y) .
\end{aligned}$$

REMARK : Note how the proof parallels that for the discrete case !

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Conditional distributions

Let X and Y be continuous random variables.

For given allowable sets I_X and I_Y (typically *intervals*), let

$$E_x = X^{-1}(I_X) \quad \text{and} \quad E_y = Y^{-1}(I_Y) ,$$

be their corresponding *events* in the sample space $\mathcal{S}$.

We have
$$P(E_x|E_y) \equiv \frac{P(E_x E_y)}{P(E_y)} .$$

The *conditional probability density function* is defined as

$$f_{X|Y}(x|y) \equiv \frac{f_{X,Y}(x,y)}{f_Y(y)} .$$

When X and Y are *independent* then

$$f_{X|Y}(x|y) \equiv \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{f_X(x)\ f_Y(y)}{f_Y(y)} = f_X(x) ,$$

(assuming $f_Y(y) \neq 0$).

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EXAMPLE : The random variables with density function

$$f_{X,Y}(x,y) = \begin{cases} e^{-x-y} & \text{for } x \geq 0 \text{ and } y \geq 0 , \\ 0 & \text{otherwise} , \end{cases}$$

have (by previous exercise) the marginal density functions

$$f_X(x) = e^{-x} , \quad f_Y(y) = e^{-y} ,$$

for $x \geq 0$ and $y \geq 0$, and zero otherwise.

Thus for such x, y we have

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{e^{-x-y}}{e^{-y}} = e^{-x} = f_X(x) ,$$

i.e., information about Y does not alter the density function of X .

Indeed, we have already seen that X and Y are *independent* .

Expectation

The *expected value* of a continuous random variable X is

$$E[X] \quad = \quad \int_{-\infty}^{\infty} x \; f_X(x) \; dx \; ,$$

which represents the *average value* of X over many trials.

The expected value of a *function of a random variable* is

$$E[g(X)] \quad = \quad \int_{-\infty}^{\infty} g(x) \; f_X(x) \; dx \; .$$

The expected value of a function of *two* random variables is

$$E[g(X,Y)] \quad = \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) \; f_{X,Y}(x,y) \; dy \; dx \; .$$

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EXAMPLE :

For the *pointer* experiment

$$f_X(x) = \begin{cases} 0\ , & x \le 0 \\ 1\ , & 0 < x \le 1 \\ 0\ , & 1 < x \end{cases}$$

we have

$$E[X] = \int_{-\infty}^{\infty} x\ f_X(x)\ dx = \int_0^1 x\ dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}\ ,$$

and

$$E[X^2] = \int_{-\infty}^{\infty} x^2\ f_X(x)\ dx = \int_0^1 x^2\ dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}\ .$$

EXAMPLE : For the joint density function

$$f_{X,Y}(x,y) = \begin{cases} e^{-x-y} & \text{for } x > 0 \text{ and } y > 0 , \\ 0 & \text{otherwise} . \end{cases}$$

we have (by previous exercise) the marginal density functions

$$f_X(x) = \begin{cases} e^{-x} & \text{for } x > 0 , \\ 0 & \text{otherwise} , \end{cases} \quad \text{and} \quad f_Y(y) = \begin{cases} e^{-y} & \text{for } y > 0 , \\ 0 & \text{otherwise} . \end{cases}$$

Thus $E[X] = \int_0^\infty x\, e^{-x}\, dx = -[(x+1)e^{-x}]\Big|_0^\infty = 1 .$ (Check !)

Similarly $E[Y] = \int_0^\infty y\, e^{-y}\, dy = 1 ,$

and

$$E[XY] = \int_0^\infty \int_0^\infty xy\, e^{-x-y}\, dy\, dx = 1 . \quad (\text{ Check ! })$$

EXERCISE :

Prove the following for *continuous* random variables :

- $E[aX] \; = \; a\, E[X]$,
- $E[aX + b] \; = \; a\, E[X] \; + \; b$,
- $E[X + Y] \; = \; E[X] \; + \; E[Y]$,

and *compare* the proofs to those for *discrete* random variables.

EXERCISE :

A stick of length 1 is split at a randomly selected point X.
(Thus X is uniformly distributed in the interval $[0, 1]$.)
Determine the expected length of the piece containing the point $1/3$.

PROPERTY : If X and Y are *independent* then

$$E[XY] \;=\; E[X] \cdot E[Y] \,.$$

PROOF :

$$\begin{aligned} E[XY] &= \int_{\mathbb{R}} \int_{\mathbb{R}} \; x\, y\, f_{X,Y}(x,y) \; dy\, dx \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \; x\, y\, f_X(x)\, f_Y(y) \; dy\, dx \qquad \text{(by independence)} \\ &= \int_{\mathbb{R}} [\, x\, f_X(x) \; \int_{\mathbb{R}} \; y\, f_Y(y)\, dy \,]\, dx \\ &= [\int_{\mathbb{R}} \; x\, f_X(x)\, dx \,] \cdot [\int_{\mathbb{R}} \; y\, f_Y(y)\, dy \,] \\ &= E[X] \cdot E[Y] \,. \end{aligned}$$

REMARK : Note how the proof parallels that for the discrete case !

EXAMPLE : For

$$f_{X,Y}(x,y) = \begin{cases} e^{-x-y} & \text{for } x > 0 \text{ and } y > 0 \\ 0 & \text{otherwise} \end{cases},$$

we already found

$$f_X(x) = e^{-x} \quad , \quad f_Y(y) = e^{-y} \; ,$$

so that

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y) \; ,$$

i.e., X and Y are *independent* .

Indeed, we also already found that

$$E[X] = E[Y] = E[XY] = 1 \; ,$$

so that

$$E[XY] = E[X] \cdot E[Y] \; .$$

Variance

Let $\mu = E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$

Then the *variance* of the continuous random variable X is

$$Var(X) \equiv E[\,(X-\mu)^2] \equiv \int_{-\infty}^{\infty} (x-\mu)^2\, f_X(x)\, dx\ ,$$

which is the average weighted *square distance* from the mean.

As in the discrete case, we have

$$Var(X) = E[X^2 - 2\mu X + \mu^2]$$

$$= E[X^2] - 2\mu E[X] + \mu^2 = E[X^2] - \mu^2\ .$$

The *standard deviation* of X is

$$\sigma(X) \equiv \sqrt{Var(X)} = \sqrt{E[X^2] - \mu^2}\ .$$

which is the average weighted *distance* from the mean.

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EXAMPLE : For $f(x) = \begin{cases} e^{-x}\,, & x > 0\,, \\ 0\,, & x \leq 0\,, \end{cases}$

we have

$$E[X] = \mu = \int_0^\infty x\, e^{-x}\, dx = 1 \quad (\text{ already done ! })\,,$$

$$E[X^2] = \int_0^\infty x^2\, e^{-x}\, dx = -\left[(x^2+2x+2)e^{-x}\right]\Big|_0^\infty = 2\,,$$

$$Var(X) = E[X^2] - \mu^2 = 2 - 1^2 = 1\,,$$

$$\sigma(X) = \sqrt{Var(X)} = 1\,.$$

NOTE : The two integrals can be done by "*integration by parts*".

EXERCISE :

Also use the *Method of Moments* to compute $E[X]$ and $E[X^2]$.

EXERCISE : For the random variable X with density function

$$f(x) = \begin{cases} 0\ , & x \leq -1 \\ \text{c}\ , & -1 < x \leq 1 \\ 0\ , & x > 1 \end{cases}$$

- Determine the value of c
- Draw the graph of $f(x)$
- Determine the distribution function $F(x)$
- Draw the graph of $F(x)$
- Determine $E[X]$
- Compute $Var(X)$ and $\sigma(X)$
- Determine $P(X \leq -\frac{1}{2})$
- Determine $P(\mid X \mid \geq \frac{1}{2})$

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EXERCISE : For the random variable X with density function

$$f(x) = \begin{cases} \frac{3}{4}(1-x^2), & -1 < x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

- Draw the graph of $f(x)$
- Verify that $\int_{-\infty}^{\infty} f(x)\, dx = 1$
- Determine the distribution function $F(x)$
- Draw the graph of $F(x)$
- Determine $E[X]$
- Compute $Var(X)$ and $\sigma(X)$
- Determine $P(X \leq 0)$
- Compute $P(X \geq \frac{2}{3})$
- Compute $P(| X | \geq \frac{2}{3})$

EXERCISE : Recall the density function

$$f_n(x) = \begin{cases} cx^n(1 - x^n) \ , & 0 \leq x \leq 1 \\ 0 \ , & \text{otherwise} \end{cases}$$

considered earlier, where n is a positive integer, and where

$$c = \frac{(n+1)(2n+1)}{n} \ .$$

- Determine $E[X]$.
- What happens to $E[X]$ for *large* n ?
- Determine $E[X^2]$
- What happens to $E[X^2]$ for *large* n ?
- What happens to $Var(X)$ for *large* n ?

Covariance

Let X and Y be continuous random variables with *mean*

$$E[X] = \mu_X \quad , \quad E[Y] = \mu_Y \ .$$

Then the *covariance* of X and Y is

$$Cov(X,Y) \equiv E[\,(X-\mu_X)\,(Y-\mu_Y)\,]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_X)\,(y-\mu_Y)\, f_{X,Y}(x,y)\, dy\, dx \ .$$

As in the discrete case, we have

$$Cov(X,Y) = E[\,(X-\mu_X)\,(Y-\mu_Y)\,]$$

$$= E[XY - \mu_X Y - \mu_Y X + \mu_X\mu_Y]$$

$$= E[XY] - E[X]\,E[Y] \ .$$

As in the discrete case, we also have

PROPERTY 1 :

- $Var(X+Y) = Var(X) + Var(Y) + 2\,Cov(X,Y)$,

and

PROPERTY 2 : If X and Y are *independent* then

- $Cov(X,Y) = 0$,
- $Var(X+Y) = Var(X) + Var(Y)$.

NOTE :

- The proofs are identical to those for the discrete case !
- As in the discrete case, if $Cov(X,Y) = 0$ then X and Y are not necessarily independent!

EXAMPLE : For

$$f_{X,Y}(x,y) = \begin{cases} e^{-x-y} & \text{for } x > 0 \text{ and } y > 0 , \\ 0 & \text{otherwise} , \end{cases}$$

we already found

$$f_X(x) = e^{-x} \quad , \quad f_Y(y) = e^{-y} ,$$

so that

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y) ,$$

i.e., X and Y are *independent* .

Indeed, we also already found

$$E[X] = E[Y] = E[XY] = 1 ,$$

so that

$$Cov(X,Y) = E[XY] - E[X]\,E[Y] = 0 .$$

EXERCISE :

Verify the following properties :

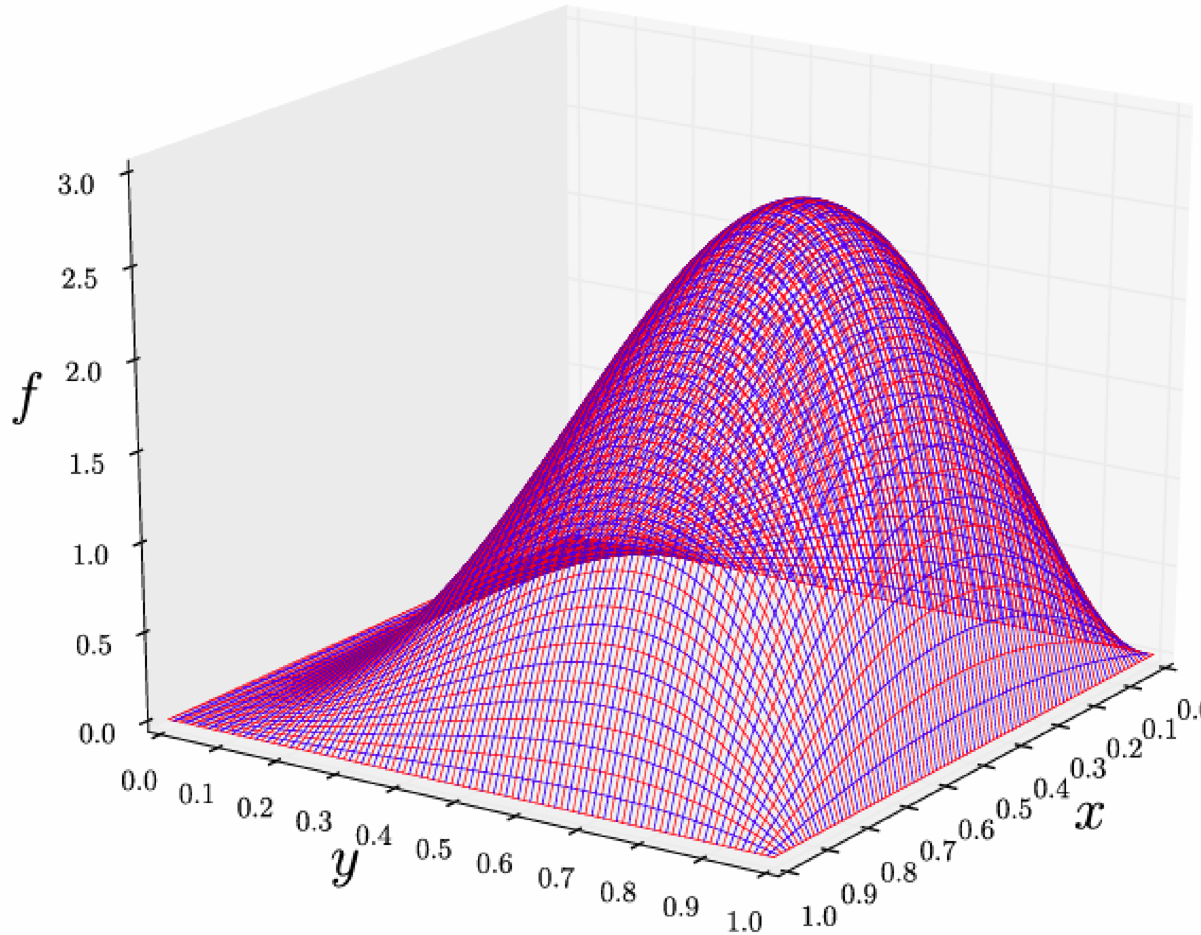
- $Var(cX+d) \;=\; c^2\, Var(X)$,
- $Cov(X,Y) \;=\; Cov(Y,X)$,
- $Cov(cX,Y) \;=\; c\, Cov(X,Y)$,
- $Cov(X,cY) \;=\; c\, Cov(X,Y)$,
- $Cov(X+Y,Z) \;=\; Cov(X,Z) \;+\; Cov(Y,Z)$,
- $Var(X+Y) \;=\; Var(X) \;+\; Var(Y) \;+\; 2\, Cov(X,Y)$.

EXERCISE :

For the random variables X , Y with *joint density function*

$$f(x,y) = \begin{cases} 45xy^2(1-x)(1-y^2) \,, & 0 \le x \le 1 \,,\ 0 \le y \le 1 \\ 0 \,, & \text{otherwise} \end{cases}$$

- Verify that $\int_0^1 \int_0^1 f(x,y)\, dy\, dx = 1$.
- Determine the *marginal density functions* $f_X(x)$ and $f_Y(y)$.
- Are X and Y *independent* ?
- What is the value of $Cov(X,Y)$?



The joint probability density function $f_{XY}(x, y)$.

Markov's inequality.

For a continuous *nonnegative* random variable X , and $c > 0$, we have

$$P(X \geq c) \quad \leq \quad \frac{E[X]}{c} .$$

PROOF :

$$E[X] \;=\; \int_0^\infty x f(x)\, dx \;=\; \int_0^c x f(x)\, dx \;+\; \int_c^\infty x f(x)\, dx$$

$$\geq \int_c^\infty x f(x)\, dx$$

$$\geq c \int_c^\infty f(x)\, dx \qquad (\text{ Why ? })$$

$$= c\, P(X \geq c) .$$

EXERCISE :

Show Markov's inequality also holds for *discrete* random variables.

Markov's inequality : For continuous *nonnegative* X , $c > 0$:

$$P(X \geq c) \quad \leq \quad \frac{E[X]}{c} \; .$$

EXAMPLE : For

$$f(x) \; = \; \begin{cases} e^{-x} & \text{for } x > 0 \; , \\ 0 & \text{otherwise} \; , \end{cases}$$

we have

$$E[X] \; = \; \int_0^\infty x \, e^{-x} \, dx \; = \; 1 \qquad (\text{ already done ! })$$

Markov's inequality gives

$$c = 1 \; : \qquad P(X \geq 1) \; \leq \; \frac{E[X]}{1} \; = \; \frac{1}{1} \; = \; 1 \;\; (\, ! \,)$$

$$c = 10 \; : \qquad P(X \geq 10) \; \leq \; \frac{E[X]}{10} \; = \; \frac{1}{10} \; = \; 0.1$$

QUESTION : Are these estimates "*sharp*" ?

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Markov's inequality gives

$$c = 1 \ : \qquad P(X \geq 1) \ \leq \ \frac{E[X]}{1} \ = \ \frac{1}{1} \ = \ 1 \ \ (\,!\,)$$

$$c = 10 \ : \qquad P(X \geq 10) \ \leq \ \frac{E[X]}{10} \ = \ \frac{1}{10} \ = \ 0.1$$

The actual values are

$$P(X \geq 1) \ = \ \int_1^{\infty} e^{-x}\, dx \ = \ e^{-1} \ \cong \ 0.37$$

$$P(X \geq 10) \ = \ \int_{10}^{\infty} e^{-x}\, dx \ = \ e^{-10} \ \cong \ 0.000045$$

EXERCISE : Suppose the score of students taking an examination is a random variable with mean 65 .

Give an upper bound on the probability that a student's score is greater than 75 .

Chebyshev's inequality: For (practically) any random variable X:

$$P(\ |X-\mu|\ \geq\ k\,\sigma\ )\ \leq\ \frac{1}{k^2}\ ,$$

where $\mu = E[X]$ is the *mean*, $\sigma = \sqrt{Var(X)}$ the *standard deviation*.

PROOF : Let $Y \equiv (X-\mu)^2$, which is nonnegative.

By Markov's inequality

$$P(Y \geq c)\ \leq\ \frac{E[Y]}{c}\ .$$

Taking $c = k^2\sigma^2$ we have

$$P(\ |X-\mu|\ \geq\ k\sigma\ )\ =\ P(\ (X-\mu)^2\ \geq\ k^2\sigma^2\ )\ =\ P(\ Y\ \geq\ k^2\sigma^2\ )$$

$$\leq\ \frac{E[\ Y\ ]}{k^2\sigma^2}\ =\ \frac{Var(X)}{k^2\sigma^2}\ =\ \frac{\sigma^2}{k^2\sigma^2}\ =\ \frac{1}{k^2}\ .$$ **QED** !

NOTE : This inequality also holds for discrete random variables.

EXAMPLE : Suppose the value of the Canadian dollar in terms of the US dollar over a certain period is a random variable X with

$$\textit{mean} \quad \mu = 0.98 \quad \text{and} \quad \textit{standard deviation} \quad \sigma = 0.05 \; .$$

What can be said of the probability that the Canadian dollar is valued

$$\text{between} \quad \$0.88\text{US} \quad \text{and} \quad \$1.08\text{US} \quad ,$$

that is,

$$\text{between} \quad \mu - 2\sigma \quad \text{and} \quad \mu + 2\sigma \quad ?$$

SOLUTION : By Chebyshev's inequality we have

$$P(\; | X - \mu | \;\; \geq \; 2 \, \sigma \;) \;\; \leq \;\; \frac{1}{2^2} \;\; = \;\; 0.25 \; .$$

Thus

$$P(\; | X - \mu | \;\; < \; 2 \, \sigma \;) \;\; > \;\; 1 \; - \; 0.25 \;\; = \;\; 0.75 \; ,$$

that is,

$$P(\; \$0.88\text{US} \; < \; \text{Can}\$ \; < \; \$1.08\text{US} \;) \;\; > \;\; 75 \, \% \; .$$

EXERCISE :

The score of students taking an examination is a random variable with mean $\mu = 65$ and standard deviation $\sigma = 5$.

- What is the probability a student scores between 55 and 75 ?
- How many students would have to take the examination so that the probability that their average grade is between 60 and 70 is at least 80% ?

HINT : Defining

$$\bar{X} = \frac{1}{n}(X_1 + X_2 + \cdots + X_n) , \quad \text{(the average grade)}$$

we have

$$\mu_{\bar{X}} = E[\bar{X}] = \frac{1}{n} n \mu = \mu = 65 ,$$

and, assuming independence,

$$Var(\bar{X}) = n \frac{\sigma^2}{n^2} = \frac{\sigma^2}{n} = \frac{25}{n} , \quad \text{and} \quad \sigma_{\bar{X}} = \frac{5}{\sqrt{n}} .$$

SUBRATA SAHA

THANK YOU

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