Recurrence Relations

Recurrence Relations

A **recurrence relation** for the sequence $\{a_n\}$ is an equation that expresses a_n is terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

A sequence is called a **solution** of a recurrence relation if it terms satisfy the recurrence relation.

Problem

A pair of newly born rabbits of opposite sexes is placed in an enclosure at the beginning of a year. Baby rabbits need one moth to grow mature; they become an adult pair on the first day of the second month. Beginning with the second month the female is pregnant, and gives exactly one birth of one pair of rabbits of opposite sexes on the first day of the third month, and gives exactly one such birth on the first day of each next month. Each new pair also gives such birth to a pair of rabbits on the first day of each month starting from the third month (from its birth). Find the number of pairs of rabbits in the enclosure after one year?

SOLVE THE PROBLEM

Some useful formulae

Ordinal number	Generating function $G(z)$	Recursive sequence $\{a_n\}$
1.	$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$	$1, 1, 1, 1, \ldots$
2.	$\frac{1}{1-cz} = \sum_{n=0}^{\infty} c^n z^n$	$1, c, c^2, c^3, \ldots$
3.	$\frac{1-z^{N+1}}{1-z} = \sum_{n=0}^{N} z^n$	$\underbrace{1, 1, 1, \ldots, 1}_{N \text{ terms}}, 0, 0, 0, \ldots$
4.	$(1+z)^N = \sum_{n=0}^{\infty} C(N,n) z^n$	$C(N,0), C(N,1), C(N,2), C(N,3), \ldots$
5.	$(1+cz)^N = \sum_{n=0}^{\infty} C(N,n) c^n z^n$	$C(N,0), C(N,1)c, C(N,2)c^2, C(N,3)c^3, \ldots$
6.	$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1) z^n$	$1, 2, 3, 4, \ldots$
7.	$\frac{1}{(1-z)^N} = \sum_{n=0}^{\infty} C(N+n-1, n) z^n$	$C(N-1,0), C(N,1), C(N+1,2), C(N+2,3), \ldots$
8.	$\frac{1}{(1-cz)^N} = \sum_{n=0}^{\infty} C(N+n-1, n) c^n z^n$	$C(N-1,0), C(N,1)c, C(N+1,2)c^2, C(N+2,3)c^3, \ldots$
9.	$\frac{1}{1-z^N} = \sum_{n=0}^{\infty} z^{N \cdot n}$	$\underbrace{1, 0, 0, \ldots, 0}_{N \text{ terms}}, 1, 0, 0, \ldots$
10.	$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$	$1, 1, \frac{1}{2!}, \frac{1}{3!}, \frac{1}{4!}, \ldots$
11.	$\ln(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n}$	$0, 1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots$

Recurrence Relations

Example:

Consider the recurrence relation

$a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$

Is the sequence $\{a_n\}$ with $a_n=3n$ a solution of this recurrence relation?

For $n \geq 2$ we see that

$2a_{n-1} - a_{n-2} = 2(3(n - 1)) - 3(n - 2) = 3n = a_n$.

Therefore, $\{a_n\}$ with $a_n=3n$ is a solution of the recurrence relation.

Recurrence Relations

Is the sequence $\{a_n\}$ with $a_n=5$ a solution of the same recurrence relation?

For $n \geq 2$ we see that
$2a_{n-1} - a_{n-2} = 2\cdot5 - 5 = 5 = a_n$.

Therefore, $\{a_n\}$ with $a_n=5$ is also a solution of the recurrence relation.

Modeling with Recurrence Relations

Example:

Someone deposits \$10,000 in a savings account at a bank yielding 5% per year with interest compounded annually. How much money will be in the account after 30 years?

Solution:

Let P_n denote the amount in the account after n years.

How can we determine P_n on the basis of P_{n-1}?

Modeling with Recurrence Relations

We can derive the following **recurrence relation**:

$P_n = P_{n-1} + 0.05P_{n-1} = 1.05P_{n-1}$.

The initial condition is $P_0 = 10{,}000$.

Then we have:

$P_1 = 1.05P_0$

$P_2 = 1.05P_1 = (1.05)^2P_0$

$P_3 = 1.05P_2 = (1.05)^3P_0$

…

$P_n = 1.05P_{n-1} = (1.05)^nP_0$

We now have a **formula** to calculate P_n for any natural number n and can avoid the iteration.

Modeling with Recurrence Relations

Let us use this formula to find P_{30} under the initial condition $P_0 = 10{,}000$:

$P_{30} = (1.05)^{30} \cdot 10{,}000 = 43{,}219.42$

After 30 years, the account contains $43,219.42.

Solving Recurrence Relations

In general, we would prefer to have an **explicit formula** to compute the value of a_n rather than conducting n iterations.

For one class of recurrence relations, we can obtain such formulas in a systematic way.

Those are the recurrence relations that express the terms of a sequence as **linear combinations** of previous terms.

Solving Recurrence Relations

Definition: A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form:

$a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}$,

Where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

A sequence satisfying such a recurrence relation is uniquely determined by the recurrence relation and the k initial conditions

$a_0 = C_0, a_1 = C_1, a_2 = C_2, \ldots, a_{k-1} = C_{k-1}$.

Solving Recurrence Relations

Examples:

The recurrence relation $P_n = (1.05)P_{n-1}$ is a linear homogeneous recurrence relation of **degree one**.

The recurrence relation $f_n = f_{n-1} + f_{n-2}$ is a linear homogeneous recurrence relation of **degree two**.

The recurrence relation $a_n = a_{n-5}$ is a linear homogeneous recurrence relation of **degree five**.

Solving Recurrence Relations

Basically, when solving such recurrence relations, we try to find solutions of the form $a_n = r^n$, where r is a constant.

$a_n = r^n$ is a solution of the recurrence relation
$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$ if and only if
$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \ldots + c_k r^{n-k}$.

Divide this equation by r^{n-k} and subtract the right-hand side from the left:

$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c_{k-1} r - c_k = 0$

This is called the **characteristic equation** of the recurrence relation.

Solving Recurrence Relations

The solutions of this equation are called the **characteristic roots** of the recurrence relation.

Let us consider linear homogeneous recurrence relations of **degree two**.

Theorem: Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

Solving Recurrence Relations

A linear homogeneous recurrence relation of degree k with constant coefficient is a recurrence relation of the form

$a_n = c_1 a_{n-1} + \ldots + c_k a_{n-k}$,
where the c_i are all real, and $c_k \neq 0$.

The solution is uniquely determined if k initial conditions $a_0 \ldots a_{k-1}$ are provided

Solving Recurrence Relations

Example: What is the solution of the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$?

Solution: The characteristic equation of the recurrence relation is $r^2 - r - 2 = 0$.

Its roots are $r = 2$ and $r = -1$.

Hence, the sequence $\{a_n\}$ is a solution to the recurrence relation if and only if:

$a_n = \alpha_1 2^n + \alpha_2(-1)^n$ for some constants α_1 and α_2.

Solving Recurrence Relations

Given the equation $a_n = \alpha_1 2^n + \alpha_2(-1)^n$ and the initial conditions $a_0 = 2$ and $a_1 = 7$, it follows that

$a_0 = 2 = \alpha_1 + \alpha_2$

$a_1 = 7 = \alpha_1 \cdot 2 + \alpha_2 \cdot (-1)$

Solving these two equations gives us
$\alpha_1 = 3$ and $\alpha_2 = -1$.

Therefore, the solution to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with

$a_n = 3 \cdot 2^n - (-1)^n$.

Solving Recurrence Relations

$a_n = r^n$ is a solution of the linear homogeneous recurrence relation

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$

if and only if

$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \dots + c_k r^{n-k}$.

Divide this equation by r^{n-k} and subtract the right-hand side from the left:

$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \dots - c_{k-1} r - c_k = 0$

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Solving Recurrence Relations

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Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

Solving Recurrence Relations

Example: Give an explicit formula for the Fibonacci numbers.

Solution: The Fibonacci numbers satisfy the recurrence relation $f_n = f_{n-1} + f_{n-2}$ with initial conditions $f_0 = 0$ and $f_1 = 1$.

The characteristic equation is $r^2 - r - 1 = 0$.

Its roots are

$$r_1 = \frac{1+\sqrt{5}}{2}, \quad r_2 = \frac{1-\sqrt{5}}{2}$$

Solving Recurrence Relations

Therefore, the Fibonacci numbers are given by

$$f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2} \right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2} \right)^n$$

for some constants α_1 and α_2.

We can determine values for these constants so that the sequence meets the conditions $f_0 = 0$ and $f_1 = 1$:

$$f_0 = \alpha_1 + \alpha_2 = 0$$

$$f_1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2} \right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2} \right) = 1$$

Generating Functions

Given a sequence <g0,g1,g2,g3,………>

the ordinary generating function is:

$$G(x) = g_0 + g_1 x + g_2 x^2 + g_3 x^3 + \ldots$$

We use a double-sided arrow to indicate the correspondence.

$$< g_0, g_1, g_2, g_3, \ldots > \quad \leftrightarrow \quad g_0 + g_1 x + g_2 x^2 + g_3 x^3 + \ldots$$

Examples

$$< 1, 1, 1, 1, \ldots > \quad \leftrightarrow \quad 1 + 1x + 1x^2 + 1x^3 + \ldots \quad = \quad \frac{1}{1-x}$$

$$< 1, -1, 1, -1, \ldots > \quad \leftrightarrow \quad 1 - 1x + 1x^2 - 1x^3 + \ldots \quad = \quad \frac{1}{1+x}$$

$$< 1, a, a^2, a^3, \ldots > \quad \leftrightarrow \quad 1 + ax + a^2x^2 + a^3x^3 + \ldots \quad = \quad \frac{1}{1-ax}$$

$$< 1, 0, 1, 0, \ldots > \quad \leftrightarrow \quad 1 + 0x + 1x^2 + 0x^3 + \ldots \quad = \quad \frac{1}{1-x^2}$$

Solving Recurrence Relations

The unique solution to this system of two equations and two variables is

$$\alpha_1 = \frac{1}{\sqrt{5}}\ ,\ \ \alpha_2 = -\frac{1}{\sqrt{5}}$$

So finally we obtained an explicit formula for the Fibonacci numbers:

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Solving Recurrence Relations

If the "largest" index term in the recurrence is a_n, and the "smallest" index term is a_{n-k}, the characteristic equation will be a polynomial of degree k. There will be a non-zero term involving x k–t whenever a_{n-t} is involved in the recurrence.

Solving Recurrence Relations

But what happens if the characteristic equation has only one root?

How can we then match our equation with the initial conditions a_0 and a_1 ?

Theorem: Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0. A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$, for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

Solving Recurrence Relations

Example: What is the solution of the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1$ and $a_1 = 6$?

Solution: The only root of $r^2 - 6r + 9 = 0$ is $r_0 = 3$.

Hence, the solution to the recurrence relation is

$a_n = \alpha_1 3^n + \alpha_2 n3^n$ for some constants α_1 and α_2.

To match the initial condition, we need

$a_0 = 1 = \alpha_1$

$a_1 = 6 = \alpha_1 \cdot 3 + \alpha_2 \cdot 3$

Solving these equations yields $\alpha_1 = 1$ and $\alpha_2 = 1$.

Consequently, the overall solution is given by

$a_n = 3^n + n3^n$.

Solve : 1 $a_n = 2 + 3a_{n-1} \quad n \geq 1, \; a_0 = 2$

Soln

Note: $g(x) = \sum_{n=0}^{\infty} a_n x^n$

$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \infty$.

$$\sum_{n=1}^{\infty} a_n x^n = 2 \sum_{n=1}^{\infty} x^n + 3 \sum_{n=1}^{\infty} a_{n-1} x^n$$

Now $\sum_{n=1}^{\infty} a_n x^n = a_1 x + a_2 x^2 + a_3 x^3 + \cdots \infty$

$= g(x) - a_0 = g(x) - 2$

$\sum_{n=1}^{\infty} a_{n-1} x^n = a_0 x + a_1 x^2 + a_2 x^3 + \cdots \infty$

$= x [a_0 + a_1 x + a_2 x^2 + \cdots \infty]$

$= x\, g(x)$

Note $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \infty$.

$\frac{1}{1-x} = 1 - x + x^2 + x^3 - \cdots \infty$.

$\sum_{n=1}^{\infty} x^n = x + x^2 + x^3 + \cdots \infty$

$= x (1 + x + x^2 + \cdots \infty)$

$= \frac{x}{1-x}$

Therefore

$g(x) - 2 = \frac{2x}{1-x} + 3x\, g(x)$

$\Rightarrow g(x)(1 - 3x) = \frac{2x}{1-x} + 2 = \frac{2}{1-x}$.

$\Rightarrow g(x) = \frac{2}{(1-3x)(1-x)}$

If we assume, $\frac{2}{(1-3x)(1-x)} = \frac{A}{1-3x} + \frac{B}{1-x}$

$\Rightarrow 2 = A(1-x) + B(1-3x)$

if $x = 1$

$\Rightarrow 2 = A \cdot 0 + B(1-3)$

$\Rightarrow \boxed{B = -1}$

if $x = 1/3$

$\Rightarrow 2 = A \cdot 2/3$

$\Rightarrow \boxed{A = 3}$

Now $g(x) = \frac{3}{1-3x} - \frac{1}{1-x}$ [Subs^{t} A & B]

$$g(x) = 3 \cdot \sum_{n=0}^{\infty} 3^n x^n - \sum_{n=0}^{\infty} x^n$$

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} 3^{n+1} \cdot x^n - \sum_{n=0}^{\infty} 1 \cdot x^n$$

$$\Rightarrow \boxed{a_n = 3^{n+1} - 1}$$

Solve : 2 $\quad a_n - 8a_{n-1} = 10^{n-1}, \quad a_0 = 1$

Sol^n

$$\sum_{n=1}^{\infty} a_n x^n - 8\sum_{n=1}^{\infty} a_{n-1} x^n = \sum_{n=1}^{\infty} 10^{n-1} x^n$$

Following Solve I

$$\sum_{n=1}^{\infty} a_n x^n = g(x) - a_0 = g(x) - 1$$

$$\sum_{n=1}^{\infty} a_{n-1} x^n = x\,g(x)$$

Now $\quad \sum_{n=1}^{\infty} 10^{n-1} x^n = x + 10x^2 + 10^2x^3 + 10^3x^4 + \cdots \infty.$

$$= x\left[1 + 10x + 10^2x^2 + 10^3x^3 + \cdots \infty\right] \quad [G.P]$$

$$= x\,\frac{1}{1-10x}$$

$$g(x) - 1 - 8x\,g(x) = \frac{x}{1-10x}$$

$$g(x)\left[1-8x\right] = \frac{x}{1-10x} + 1 = \frac{1-9x}{1-10x}$$

$$g(x) = \frac{1-9x}{(1-10x)(1-8x)}$$

Assume $\quad \dfrac{1-9x}{(1-10x)(1-8x)} = \dfrac{A}{1-10x} + \dfrac{B}{1-8x}$

$$1-9x = A(1-8x) + B(1-10x)$$

if $x = 1/8$

$\Rightarrow 1 - \frac{9}{8} = B\left(1 - \frac{10}{8}\right)$

$B = 1/2$

if $x = \frac{1}{10}$

$\Rightarrow 1 - \frac{9}{10} = A\left(1 - \frac{8}{10}\right)$

$A = 1/2$

$$g(x) = \frac{1}{2}\frac{1}{1-10x} + \frac{1}{2}\frac{1}{1-8x}$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n x^n = \frac{1}{2}\sum_{n=0}^{\infty} 10^n x^n + \frac{1}{2}\sum_{n=0}^{\infty} 8^n x^n$$

$$\Rightarrow \boxed{a_n = \frac{1}{2}10^n + \frac{1}{2}8^n}$$

Solve 3 $\quad a_{n+2} - 5a_{n+1} + 6a_n = 2 \quad a_0 = 1 \quad a_1 = 2$

Sol^n

$$\sum_{n=0}^{\alpha} a_{n+2}x^n - 5\sum_{n=0}^{\alpha} a_{n+1}x^n + 6\sum_{n=0}^{\alpha} a_n x^n = 2\sum_{n=0}^{\alpha} x^n$$

Now

$$\sum_{n=0}^{\alpha} a_{n+2}x^n = a_2 + a_3x + a_4x^2 + \cdots \alpha.$$

$$= \frac{1}{x^2}\left[a_2x^2 + a_3x^3 + a_4x^4 + \cdots \alpha\right]$$

$$= \frac{1}{x^2}\left[g(x) - a_0 - a_1x\right]$$

$$= \frac{1}{x^2}\left[g(x) - 1 - 2x\right]$$

$$\sum_{n=0}^{\alpha} a_{n+1}x^n = a_1 + a_2x + a_3x^2 + a_4x^3 + \cdots \alpha$$

$$= \frac{1}{x}\left[a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots \alpha\right]$$

$$= \frac{1}{x}\left[g(x) - a_0\right]$$

$$= \frac{1}{x}\left[g(x) - 1\right]$$

$$\Rightarrow \frac{1}{x^2}\left[g(x) - 1 - 2x\right] - \frac{5}{x}\left[g(x) - 1\right] + 6g(x) = \frac{2}{1-x}$$

$$\Rightarrow g(x)\left[\frac{1}{x^2} - \frac{5}{x} + 6\right] = \frac{2}{1-x} + \frac{1}{x^2} + \frac{2x}{x^2} - \frac{5}{x}$$

$$g(x) = \frac{1 - 4x + 5x^2}{(1-x)(1-3x)(1-2x)} \quad \left[\because 1 - 5x + 6x^2 = (1-2x)(1-3x)\right]$$

Assume

$$\frac{1-4x+5x^2}{(1-x)(1-2x)(1-3x)} = \frac{A}{1-x} + \frac{B}{1-2x} + \frac{C}{1-3x}$$

$$\Rightarrow 1-4x+5x^2 = A(1-2x)(1-3x) + B(1-x)(1-3x) + C(1-x)(1-2x)$$

$x=1$

$2 = A(-1)\cdot(-2)$

$A = 1$

$x = \frac{1}{2}$

$1-2+\frac{5}{4} = B(\frac{1}{2})\cdot(-\frac{1}{2})$

$\Rightarrow B = -1$

$x = \frac{1}{3}$

$1-\frac{4}{3}+\frac{5}{9} = C\cdot\frac{2}{3}\cdot\frac{1}{3}$

$\Rightarrow C = 1$

$$g(x) = \frac{1}{1-x} - \frac{1}{1-2x} + \frac{1}{1-3x}$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} x^n - \sum_{n=0}^{\infty} 2^n x^n + \sum_{n=0}^{\infty} 3^n x^n$$

$$\Rightarrow \boxed{a_n = 1 - 2^n + 3^n}$$

Please follow PAL & DAS

Ex-4

$$\boxed{a_n - 5a_{n-1} + 6a_{n-2} = 2^n + n}$$