Set Theory

Actually, you will see that logic and set theory are very closely related.

Set Theory

- Set: Collection of objects (called elements)
- $a \in A$ "a is an element of A"
  "a is a member of A"
- $a \notin A$ "a is not an element of A"
- $A = \{a_1, a_2, \ldots, a_n\}$ "A contains $a_1, \ldots, a_n$"
- Order of elements is insignificant
- It does not matter how often the same element is listed (repetition doesn't count).

Set Equality

Sets A and B are equal if and only if they contain exactly the same elements.

Examples:

- A = {9, 2, 7, -3}, B = {7, 9, -3, 2} : A = B
- A = {dog, cat, horse},
  B = {cat, horse, squirrel, dog} : A ≠ B
- A = {dog, cat, horse},
  B = {cat, horse, dog, dog} : A = B

Examples for Sets

"Standard" Sets:

- Natural numbers **N** = {0, 1, 2, 3, …}
- Integers **Z** = {…, -2, -1, 0, 1, 2, …}
- Positive Integers $\mathbf{Z}^+$ = {1, 2, 3, 4, …}
- Real Numbers **R** = {47.3, -12, π, …}
- Rational Numbers **Q** = {1.5, 2.6, -3.8, 15, …}

(correct definitions will follow)

Examples for Sets

- $A = \varnothing$ "empty set/null set"
- $A = \{z\}$ Note: $z \in A$, but $z \neq \{z\}$
- $A = \{\{b, c\}, \{c, x, d\}\}$ set of sets
- $A = \{\{x, y\}\}$ Note: $\{x, y\} \in A$, but $\{x, y\} \neq \{\{x, y\}\}$
- $A = \{x \mid P(x)\}$ "set of all x such that P(x)"

  P(x) is the membership function of set *A*

  $\forall x\ (P(x) \rightarrow x \in A)$
- $A = \{x \mid x \in \mathbf{N} \land x > 7\} = \{8, 9, 10, \ldots\}$
  "set builder notation"

Examples for Sets

We are now able to define the set of rational numbers Q:

$\mathbf{Q} = \{a/b \mid a \in \mathbf{Z} \wedge b \in \mathbf{Z}^+\}$, or

$\mathbf{Q} = \{a/b \mid a \in \mathbf{Z} \wedge b \in \mathbf{Z} \wedge b \neq 0\}$

And how about the set of real numbers R?

$\mathbf{R} = \{r \mid r \text{ is a real number}\}$

That is the best we can do. It can neither be defined by enumeration nor builder function.

Subsets

$A \subseteq B$ "A is a subset of B"

$A \subseteq B$ if and only if every element of A is also an element of B.

We can completely formalize this:

$A \subseteq B \Leftrightarrow \forall x\, (x \in A \rightarrow x \in B)$

Examples:

A = {3, 9}, B = {5, 9, 1, 3}, $A \subseteq B$? true

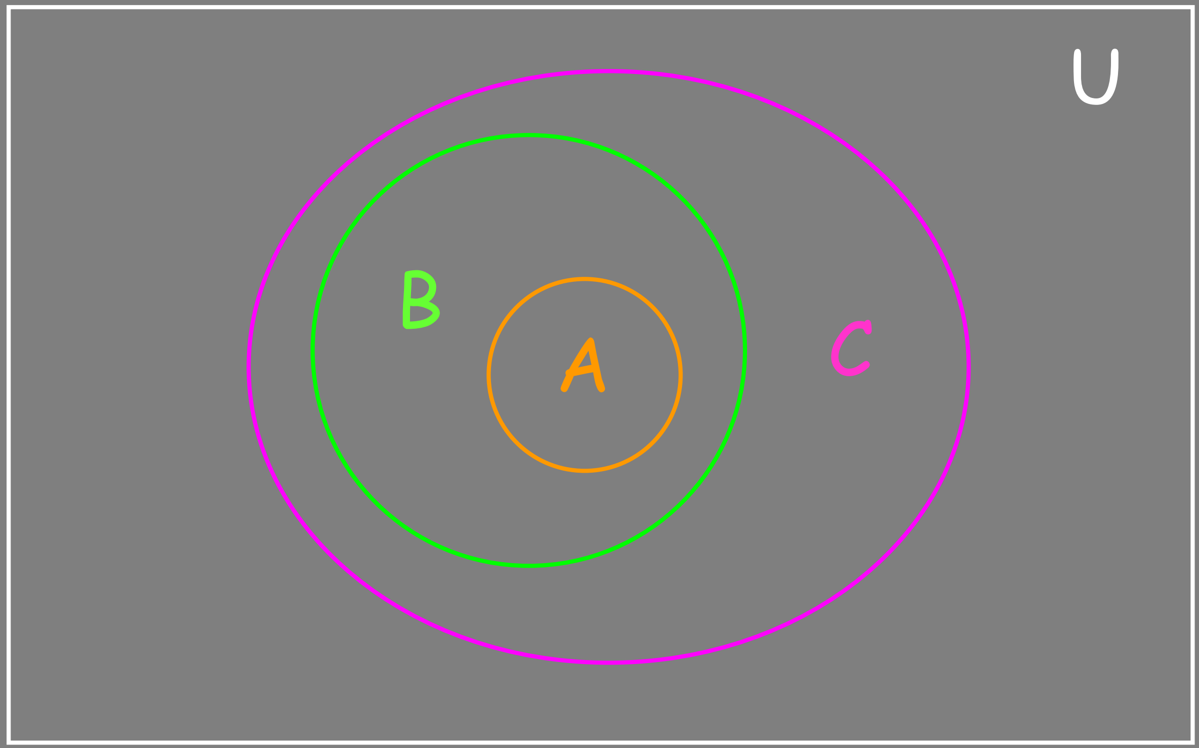
A = {3, 3, 3, 9}, B = {5, 9, 1, 3}, $A \subseteq B$? true

A = {1, 2, 3}, B = {2, 3, 4}, $A \subseteq B$? false

Subsets

Useful rules:

- $A = B \Leftrightarrow (A \subseteq B) \wedge (B \subseteq A)$
- $(A \subseteq B) \wedge (B \subseteq C) \Rightarrow A \subseteq C$ (see Venn Diagram)



Subsets

Useful rules:

- $\emptyset \subseteq A$ for any set A
  (but $\emptyset \in A$ may not hold for any set A)
- $A \subseteq A$ for any set A

Proper subsets:

$A \subset B$ "A is a proper subset of B"

$A \subset B \Leftrightarrow \forall x\,(x \in A \rightarrow x \in B) \land \exists x\,(x \in B \land x \notin A)$

or

$A \subset B \Leftrightarrow \forall x\,(x \in A \rightarrow x \in B) \land \neg\forall x\,(x \in B \rightarrow x \in A)$

Cardinality of Sets

If a set S contains n distinct elements, $n \in \mathbf{N}$, we call S a finite set with cardinality n.

Examples:

A = {Mercedes, BMW, Porsche},	$\|A\| = 3$
$B = \{1, \{2, 3\}, \{4, 5\}, 6\}$	$\|B\| = 4$
$C = \emptyset$	$\|C\| = 0$
$D = \{ x \in \mathbf{N} \mid x \leq 7000 \}$	$\|D\| = 7001$
$E = \{ x \in \mathbf{N} \mid x \geq 7000 \}$	E is infinite!

The Power Set

$P(A)$ "power set of A" (also written as 2^A)

$P(A) = \{B \mid B \subseteq A\}$ (contains all subsets of A)

Examples:

$A = \{x, y, z\}$

$P(A) = \{\emptyset, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\}\}$

$A = \emptyset$

$P(A) = \{\emptyset\}$

Note: $|A| = 0$, $|P(A)| = 1$

The Power Set

Cardinality of power sets: $| P(A) | = 2^{|A|}$

- Imagine each element in A has an "on/off" switch
- Each possible switch configuration in A corresponds to one subset of A, thus one element in $P(A)$

A	1	2	3	4	5	6	7	8
x	x	x	x	x	x	x	x	x
y	y	y	y	y	y	y	y	y
z	z	z	z	z	z	z	z	z

- For 3 elements in A, there are $2 \times 2 \times 2 = 8$ elements in $P(A)$

Cartesian Product

The ordered n-tuple $(a_1, a_2, a_3, \ldots, a_n)$ is an ordered collection of n objects.

Two ordered n-tuples $(a_1, a_2, a_3, \ldots, a_n)$ and $(b_1, b_2, b_3, \ldots, b_n)$ are equal if and only if they contain exactly the same elements in the same order, i.e. $a_i = b_i$ for $1 \leq i \leq n$.

The Cartesian product of two sets is defined as:

$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$

Cartesian Product

Example:

$A = \{good, bad\}$, $B = \{student, prof\}$

$A \times B = \{$ (good, student), (good, prof), (bad, student), (bad, prof) $\}$

$B \times A = \{$ (student, good), (prof, good), (student, bad), (prof, bad) $\}$

Example: $A = \{x, y\}$, $B = \{a, b, c\}$

$A \times B = \{(x, a), (x, b), (x, c), (y, a), (y, b), (y, c)\}$

Cartesian Product

Note that:

- $A \times \emptyset = \emptyset$
- $\emptyset \times A = \emptyset$
- For non-empty sets A and B: $A \neq B \Leftrightarrow A \times B \neq B \times A$
- $|A \times B| = |A| \cdot |B|$

The Cartesian product of two or more sets is defined as:

$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } 1 \leq i \leq n\}$

Set Operations

Union: $A \cup B = \{x \mid x \in A \vee x \in B\}$

Example: $A = \{a, b\}$, $B = \{b, c, d\}$

$A \cup B = \{a, b, c, d\}$

Intersection: $A \cap B = \{x \mid x \in A \wedge x \in B\}$

Example: $A = \{a, b\}$, $B = \{b, c, d\}$

$A \cap B = \{b\}$

Cardinality: $|A \cup B| = |A| + |B| - |A \cap B|$

Set Operations

Two sets are called disjoint if their intersection is empty, that is, they share no elements:

$A \cap B = \emptyset$

The difference between two sets A and B contains exactly those elements of A that are not in B:

$A - B = \{x \mid x \in A \wedge x \notin B\}$

Example: $A = \{a, b\}$, $B = \{b, c, d\}$, $A - B = \{a\}$

Cardinality: $|A - B| = |A| - |A \cap B|$

Set Operations

The complement of a set A contains exactly those elements under consideration that are not in A: denoted A^c (or $\overline{A}$ as in the text)

$A^c = U - A$

Example: $U = \mathbf{N}$, $B = \{250, 251, 252, \ldots\}$

$B^c = \{0, 1, 2, \ldots, 248, 249\}$

Logical Equivalence

Equivalence laws

- Identity laws, $P \wedge T \equiv P$,
- Domination laws, $P \wedge F \equiv F$,
- Idempotent laws, $P \wedge P \equiv P$,
- Double negation law, $\neg(\neg P) \equiv P$
- Commutative laws, $P \wedge Q \equiv Q \wedge P$,
- Associative laws, $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$,
- Distributive laws, $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$,
- De Morgan's laws, $\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$
- Law with implication $P \rightarrow Q \equiv \neg P \vee Q$

Set Identity

Table 1 in Section 1.7 shows many useful equations

- Identity laws, $A \cup \emptyset = A$, $A \cap U = A$
- Domination laws, $A \cup U = U$, $A \cap \emptyset = \emptyset$
- Idempotent laws, $A \cup A = A$, $A \cap A = A$
- Complementation law, $(A^c)^c = A$
- Commutative laws, $A \cup B = B \cup A$, $A \cap B = B \cap A$
- Associative laws, $A \cup (B \cup C) = (A \cup B) \cup C$, …
- Distributive laws, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, …
- De Morgan's laws, $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$
- Absorption laws, $A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$
- Complement laws, $A \cup A^c = U$, $A \cap A^c = \emptyset$

Set Identity

How can we prove $A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$?

Method I: logical equivalent

$x\in A\cup(B\cap C)$

$\Leftrightarrow x\in A \vee x\in(B\cap C)$

$\Leftrightarrow x\in A \vee (x\in B \wedge x\in C)$

$\Leftrightarrow (x\in A \vee x\in B) \wedge (x\in A \vee x\in C)$ (distributive law)

$\Leftrightarrow x\in(A\cup B) \wedge x\in(A\cup C)$

$\Leftrightarrow x\in(A\cup B)\cap(A\cup C)$

Every logical expression can be transformed into an equivalent expression in set theory and vice versa.

Set Operations

Method II: Membership table

1 means "x is an element of this set"

0 means "x is not an element of this set"

A	B	C	$B \cap C$	$A \cup (B \cap C)$	$A \cup B$	$A \cup C$	$(A \cup B) \cap (A \cup C)$
0	0	0	0	0	0	0	0
0	0	1	0	0	0	1	0
0	1	0	0	0	1	0	0
0	1	1	1	1	1	1	1
1	0	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	1	0	0	1	1	1	1
1	1	1	1	1	1	1	1

… and the following mathematical appetizer is about…

Functions

Functions

A function f from a set A to a set B is an assignment of exactly one element of B to each element of A.

We write

f(a) = b

if b is the unique element of B assigned by the function f to the element a of A.

If f is a function from A to B, we write

$f: A \rightarrow B$

(note: Here, "$\rightarrow$" has nothing to do with if… then)

Functions

If $f: A \to B$, we say that A is the domain of f and B is the codomain of f.

If f(a) = b, we say that b is the image of a and a is the pre-image of b.

The range of $f: A \to B$ is the set of all images of all elements of A.

We say that $f: A \to B$ maps A to B.

Functions

Let us take a look at the function $f: P \rightarrow C$ with

$P = \{Linda, Max, Kathy, Peter\}$

$C = \{Boston, New York, Hong Kong, Moscow\}$

f(Linda) = Moscow

f(Max) = Boston

f(Kathy) = Hong Kong

f(Peter) = New York

Here, the range of f is C.

Functions

Let us re-specify f as follows:

f(Linda) = Moscow
f(Max) = Boston
f(Kathy) = Hong Kong
f(Peter) = Boston

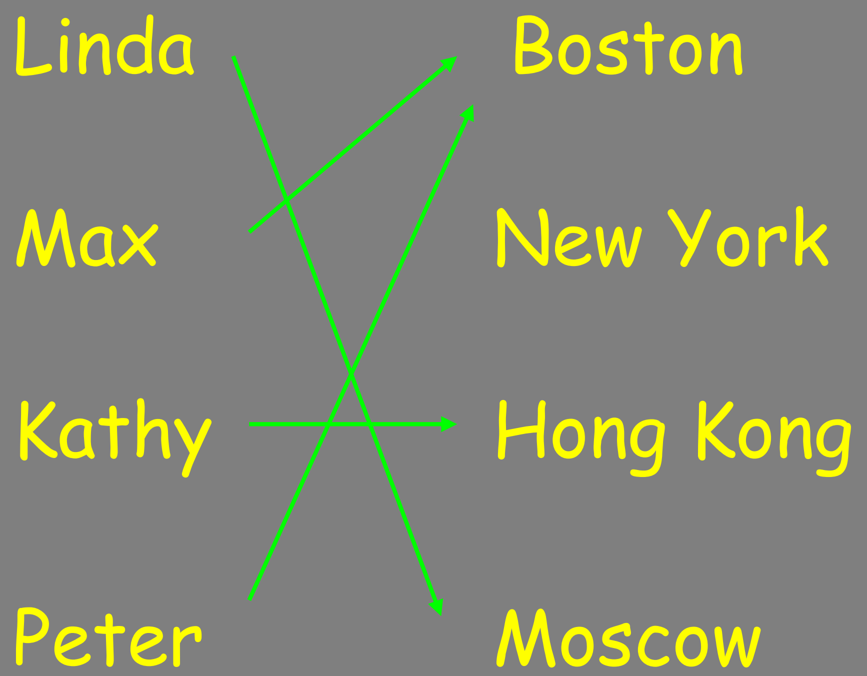
Is f still a function? yes

What is its range? {Moscow, Boston, Hong Kong}

Functions

Other ways to represent f:

x	f(x)
Linda	Moscow
Max	Boston
Kathy	Hong Kong
Peter	Boston



Functions

If the domain of our function f is large, it is convenient to specify f with a formula, e.g.:

$f{:}\mathbf{R}\rightarrow\mathbf{R}$

$f(x) = 2x$

This leads to:

$f(1) = 2$

$f(3) = 6$

$f(-3) = -6$

…

Functions

Let f_1 and f_2 be functions from *A* to **R**.

Then the sum and the product of f_1 and f_2 are also functions from *A* to **R** defined by:

$(f_1 + f_2)(x) = f_1(x) + f_2(x)$

$(f_1 f_2)(x) = f_1(x)\, f_2(x)$

Example:

$f_1(x) = 3x, \quad f_2(x) = x + 5$

$(f_1 + f_2)(x) = f_1(x) + f_2(x) = 3x + x + 5 = 4x + 5$

$(f_1 f_2)(x) = f_1(x)\, f_2(x) = 3x\,(x + 5) = 3x^2 + 15x$

Functions

We already know that the range of a function $f:A \rightarrow B$ is the set of all images of elements $a \in A$.

If we only regard a subset $S \subseteq A$, the set of all images of elements $s \in S$ is called the image of S.

We denote the image of S by f(S):

$f(S) = \{f(s) \mid s \in S\}$

Functions

Let us look at the following well-known function:

f(Linda) = Moscow

f(Max) = Boston

f(Kathy) = Hong Kong

f(Peter) = Boston

What is the image of S = {Linda, Max} ?

f(S) = {Moscow, Boston}

What is the image of S = {Max, Peter} ?

f(S) = {Boston}

Properties of Functions

A function $f:A \to B$ is said to be one-to-one (or injective), if and only if

$$\forall x, y \in A \ (f(x) = f(y) \to x = y)$$

In other words: f is one-to-one if and only if it does not map two distinct elements of A onto the same element of B.

Properties of Functions

And again…

f(Linda) = Moscow	g(Linda) = Moscow
f(Max) = Boston	g(Max) = Boston
f(Kathy) = Hong Kong	g(Kathy) = Hong Kong
f(Peter) = Boston	g(Peter) = New York

Is f one-to-one?

No, Max and Peter are mapped onto the same element of the image.

Is g one-to-one?

Yes, each element is assigned a unique element of the image.

Properties of Functions

How can we prove that a function f is one-to-one?

Whenever you want to prove something, first take a look at the relevant definition(s):

$\forall x, y \in A \ (f(x) = f(y) \rightarrow x = y)$

Example:

$f: \mathbf{R} \rightarrow \mathbf{R}$

$f(x) = x^2$

Disproof by counterexample:

$f(3) = f(-3)$, but $3 \neq -3$, so f is not one-to-one.

Properties of Functions

… and yet another example:

$f: \mathbf{R} \to \mathbf{R}$

$f(x) = 3x$

One-to-one: $\forall x, y \in A\ (f(x) = f(y) \to x = y)$

To show: $f(x) \neq f(y)$ whenever $x \neq y$ (indirect proof)

$x \neq y$

$\Leftrightarrow 3x \neq 3y$

$\Leftrightarrow f(x) \neq f(y)$,

so if $x \neq y$, then $f(x) \neq f(y)$, that is, f is one-to-one.

Properties of Functions

A function $f:A \to B$ with $A,B \subseteq R$ is called strictly increasing, if

$\forall x,y \in A\ (x < y \to f(x) < f(y))$,

and strictly decreasing, if

$\forall x,y \in A\ (x < y \to f(x) > f(y))$.

Obviously, a function that is either strictly increasing or strictly decreasing is one-to-one.

Properties of Functions

A function $f:A \to B$ is called onto, or surjective, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.

In other words, f is onto if and only if its range is its entire codomain.

A function $f: A \to B$ is a one-to-one correspondence, or a bijection, if and only if it is both one-to-one and onto.

Obviously, if f is a bijection and A and B are finite sets, then $|A| = |B|$.

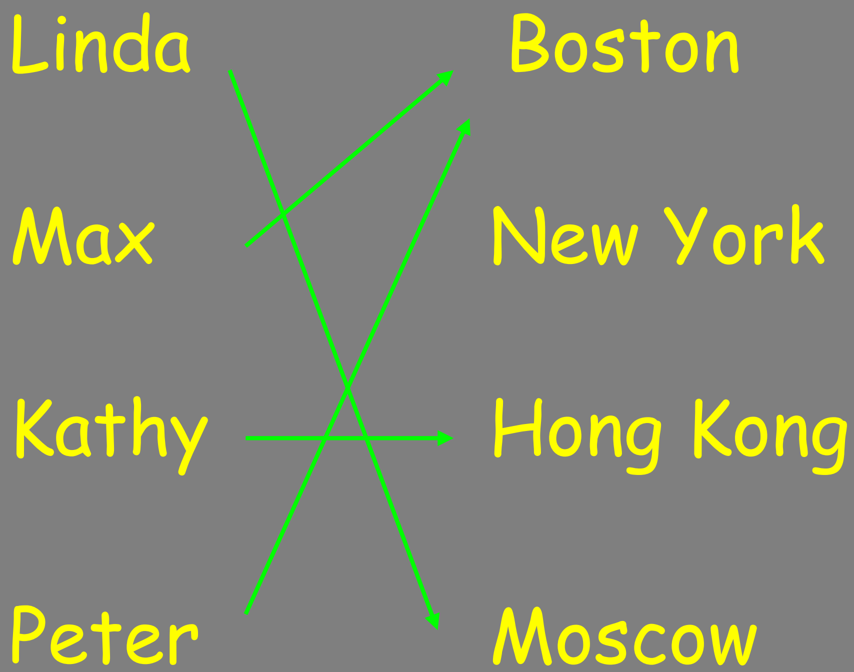
Properties of Functions

Examples:

In the following examples, we use the arrow representation to illustrate functions $f:A \rightarrow B$.

In each example, the complete sets A and B are shown.

Properties of Functions



Is f injective?

No.

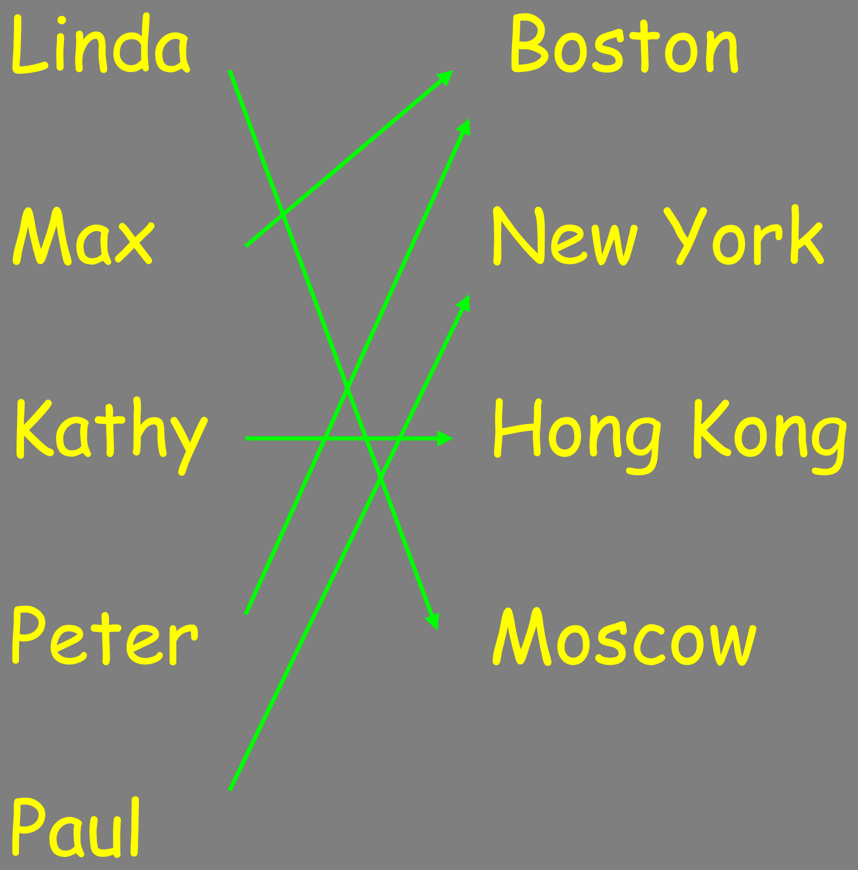
Is f surjective?

No.

Is f bijective?

No.

Properties of Functions



Is f injective?

No.

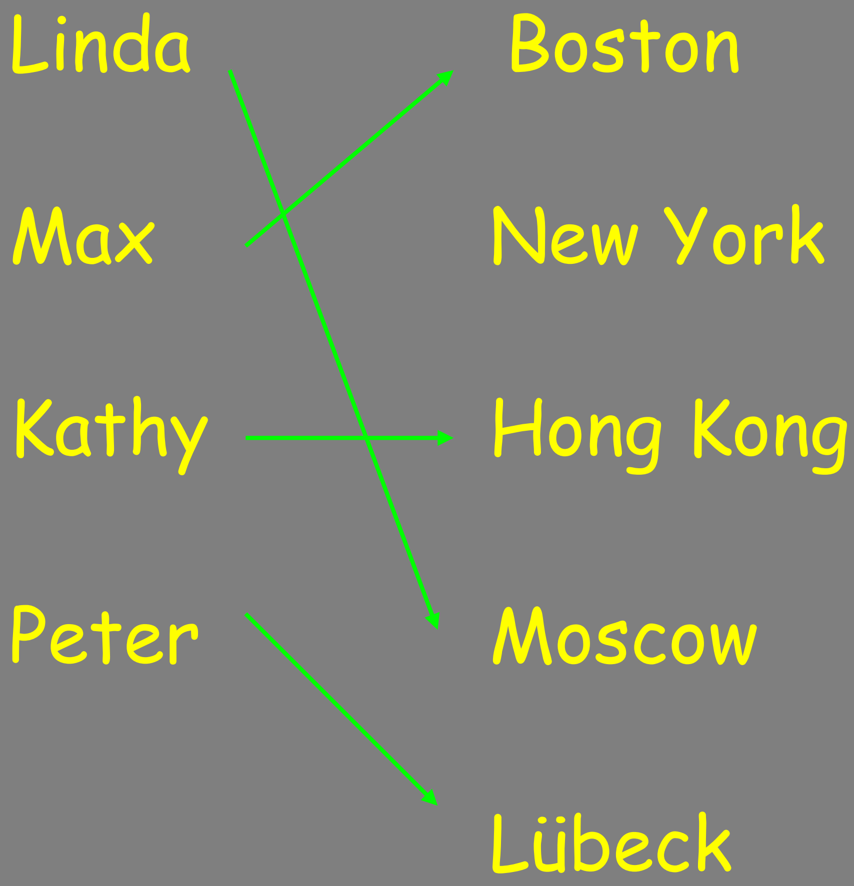
Is f surjective?

Yes.

Is f bijective?

No.

Properties of Functions



Is f injective?

Yes.

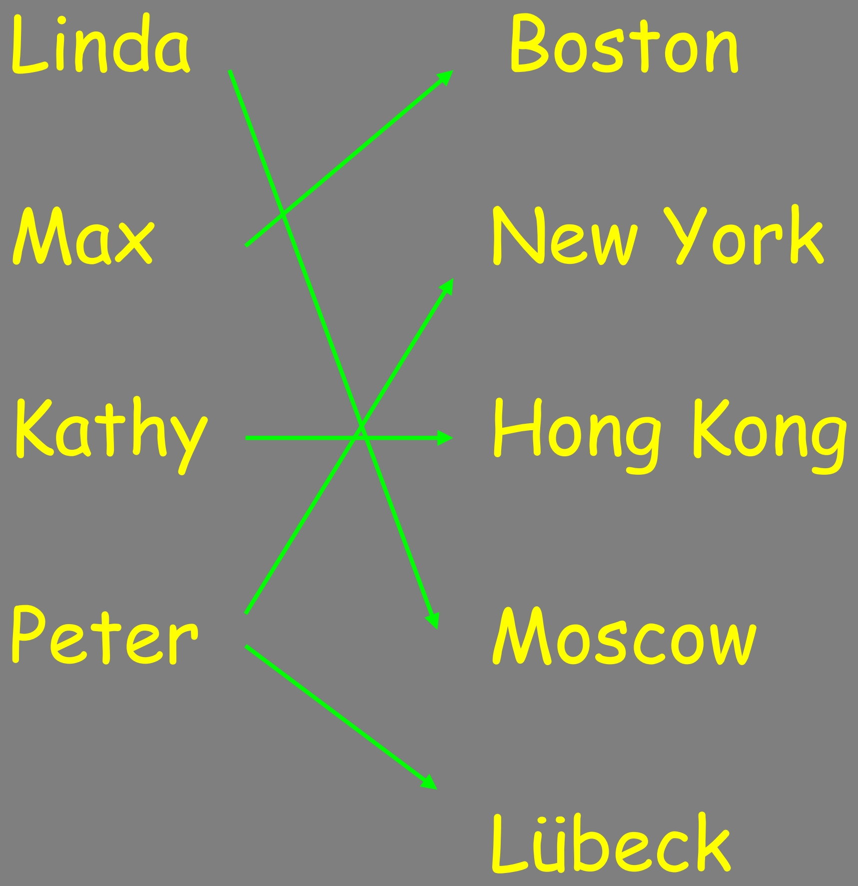
Is f surjective?

No.

Is f bijective?

No.

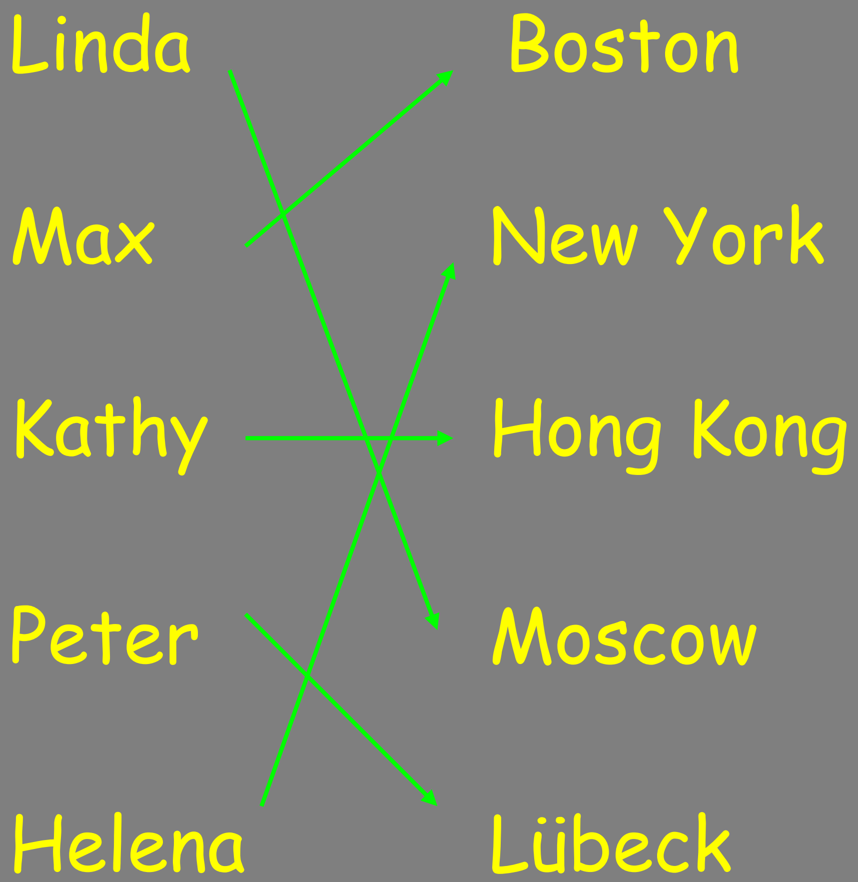
Properties of Functions



Is f injective?

No! f is not even a function!

Properties of Functions



Is f injective?

Yes.

Is f surjective?

Yes.

Is f bijective?

Yes.

Inversion

An interesting property of bijections is that they have an **inverse function**.

The **inverse function** of the bijection $f:A \rightarrow B$ is the function $f^{-1}:B \rightarrow A$ with

$f^{-1}(b) = a$ whenever $f(a) = b$.

Inversion

Example:

$f(Linda) = Moscow$
$f(Max) = Boston$
$f(Kathy) = Hong Kong$
$f(Peter) = Lübeck$
$f(Helena) = New York$

Clearly, f is bijective.

The inverse function f^{-1} is given by:

$f^{-1}(Moscow) = Linda$
$f^{-1}(Boston) = Max$
$f^{-1}(Hong Kong) = Kathy$
$f^{-1}(Lübeck) = Peter$
$f^{-1}(New York) = Helena$

Inversion is only possible for bijections (= invertible functions)

Composition

The **composition** of two functions $g{:}A \rightarrow B$ and $f{:}B \rightarrow C$, denoted by $f \circ g$, is defined by

$(f \circ g)(a) = f(g(a))$

This means that

- **first**, function g is applied to element $a \in A$, mapping it onto an element of B,
- **then**, function f is applied to this element of B, mapping it onto an element of C.
- **Therefore**, the composite function maps from A to C.

Composition

Example:

$f(x) = 7x - 4$, $g(x) = 3x$,

$f:\mathbf{R}\to\mathbf{R}$, $g:\mathbf{R}\to\mathbf{R}$

$(f\circ g)(5) = f(g(5)) = f(15) = 105 - 4 = 101$

$(f\circ g)(x) = f(g(x)) = f(3x) = 21x - 4$

Composition

Composition of a function and its inverse:

$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x$

The composition of a function and its inverse is the **identity function** $i(x) = x$.

Graphs

The **graph** of a function $f:A \rightarrow B$ is the set of ordered pairs $\{(a, b) \mid a \in A \text{ and } f(a) = b\}$.

The graph is a subset of $A \times B$ that can be used to visualize f in a two-dimensional coordinate system.

You Never Escape Your…

Relations

Relations

If we want to describe a relationship between elements of two sets A and B, we can use **ordered pairs** with their first element taken from A and their second element taken from B.

Since this is a relation between **two sets**, it is called a **binary relation**.

Definition: Let A and B be sets. A binary relation from A to B is a subset of $A \times B$.

In other words, for a binary relation R we have $R \subseteq A \times B$. We use the notation aRb to denote that $(a, b) \in R$ and $a\underline{R}b$ to denote that $(a, b) \notin R$.

Relations

When (a, b) belongs to R, a is said to be **related** to b by R.

Example: Let P be a set of people, C be a set of cars, and D be the relation describing which person drives which car(s).

P = {Carl, Suzanne, Peter, Carla},

C = {Mercedes, BMW, tricycle}

D = {(Carl, Mercedes), (Suzanne, Mercedes), (Suzanne, BMW), (Peter, tricycle)}

This means that Carl drives a Mercedes, Suzanne drives a Mercedes and a BMW, Peter drives a tricycle, and Carla does not drive any of these vehicles.

Functions as Relations

You might remember that a **function** f from a set A to a set B assigns a unique element of B to each element of A.

The **graph** of f is the set of ordered pairs (a, b) such that b = f(a).

Since the graph of f is a subset of $A \times B$, it is a **relation** from A to B.

Moreover, for each element **a** of A, there is exactly one ordered pair in the graph that has **a** as its first element.

Functions as Relations

Conversely, if R is a relation from A to B such that every element in A is the first element of exactly one ordered pair of R, then a function can be defined with R as its graph.

This is done by assigning to an element $a \in A$ the unique element $b \in B$ such that $(a, b) \in R$.

Relations on a Set

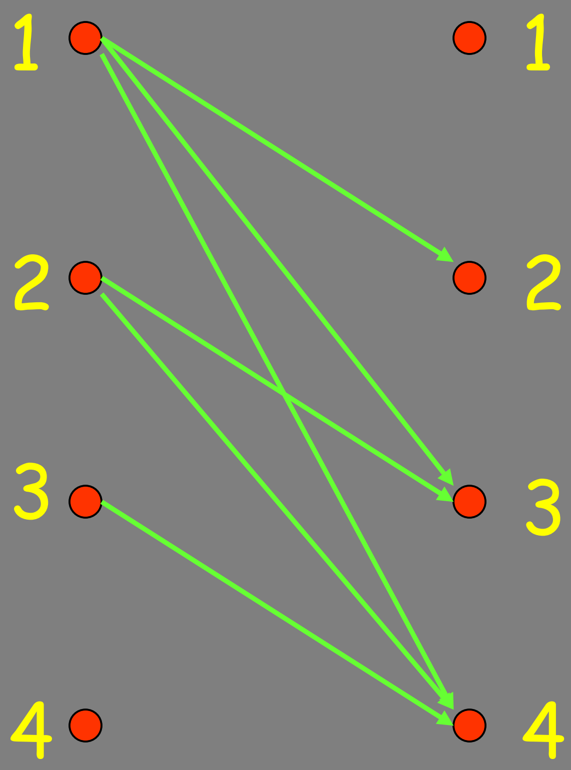
Definition: A relation on the set *A* is a relation from *A* to *A*.

In other words, a relation on the set *A* is a subset of $A \times A$.

Example: Let *A* = {1, 2, 3, 4}. Which ordered pairs are in the relation R = {(a, b) | a < b} ?

Relations on a Set

Solution: R = {(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)}



R	1	2	3	4
1		X	X	X
2			X	X
3				X
4				

Relations on a Set

How many different relations can we define on a set A with n elements?

A relation on a set A is a subset of $A \times A$.

How many elements are in $A \times A$?

There are n^2 elements in $A \times A$, so how many subsets (= relations on A) does $A \times A$ have?

The number of subsets that we can form out of a set with m elements is 2^m. Therefore, 2^{n^2} subsets can be formed out of $A \times A$.

Answer: We can define 2^{n^2} different relations on A.

Properties of Relations

We will now look at some useful ways to classify relations.

Definition: A relation R on a set A is called **reflexive** if $(a, a) \in R$ for every element $a \in A$.

Are the following relations on {1, 2, 3, 4} reflexive?

R = {(1, 1), (1, 2), (2, 3), (3, 3), (4, 4)} No.

R = {(1, 1), (2, 2), (2, 3), (3, 3), (4, 4)} Yes.

R = {(1, 1), (2, 2), (3, 3)} No.

Definition: A relation on a set A is called **irreflexive** if $(a, a) \notin R$ for every element $a \in A$.

Properties of Relations

Definitions:

A relation R on a set A is called **symmetric** if (b, a)$\in$R whenever (a, b)$\in$R for all a, b$\in$A.

A relation R on a set A is called **antisymmetric** if a = b whenever (a, b)$\in$R and (b, a)$\in$R.

A relation R on a set A is called **asymmetric** if (a, b)$\in$R implies that (b, a)$\notin$R for all a, b$\in$A.

Properties of Relations

Are the following relations on {1, 2, 3, 4} symmetric, antisymmetric, or asymmetric?

R = {(1, 1), (1, 2), (2, 1), (3, 3), (4, 4)} symmetric

R = {(1, 1)} sym. and antisym.

R = {(1, 3), (3, 2), (2, 1)} antisym. and asym.

R = {(4, 4), (3, 3), (1, 4)} antisym.

Properties of Relations

Definition: A relation R on a set A is called **transitive** if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ for $a, b, c \in A$.

Are the following relations on {1, 2, 3, 4} transitive?

R = {(1, 1), (1, 2), (2, 2), (2, 1), (3, 3)} Yes.

R = {(1, 3), (3, 2), (2, 1)} No.

R = {(2, 4), (4, 3), (2, 3), (4, 1)} No.

Counting Relations

Example: How many different reflexive relations can be defined on a set A containing n elements?

Solution: Relations on R are subsets of $A \times A$, which contains n^2 elements.

Therefore, different relations on A can be generated by choosing different subsets out of these n^2 elements, so there are 2^{n^2} relations.

A **reflexive** relation, however, **must** contain the n elements (a, a) for every $a \in A$.

Consequently, we can only choose among $n^2 - n = n(n - 1)$ elements to generate reflexive relations, so there are $2^{n(n-1)}$ of them.

Combining Relations

Relations are sets, and therefore, we can apply the usual **set operations** to them.

If we have two relations R_1 and R_2, and both of them are from a set A to a set B, then we can combine them to $R_1 \cup R_2$, $R_1 \cap R_2$, or $R_1 - R_2$.

In each case, the result will be **another relation from *A* to B**.

Combining Relations

… and there is another important way to combine relations.

Definition: Let R be a relation from a set A to a set B and S a relation from B to a set C. The **composite** of R and S is the relation consisting of ordered pairs (a, c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by **$S \circ R$**.

In other words, if relation R contains a pair (a, b) and relation S contains a pair (b, c), then $S \circ R$ contains a pair (a, c).

Combining Relations

Example: Let D and S be relations on A = {1, 2, 3, 4}.

D = {(a, b) | b = 5 - a}   "b equals (5 – a)"

S = {(a, b) | a < b}   "a is smaller than b"

D = {(1, 4), (2, 3), (3, 2), (4, 1)}

S = {(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)}

S∘D = { (2, 4), (3, 3), (3, 4), (4, 2), (4, 3), (4, 4)}

D maps an element a to the element (5 – a), and afterwards S maps (5 – a) to all elements larger than (5 – a), resulting in **S∘D = {(a,b) | b > 5 – a}** or **S∘D = {(a,b) | a + b > 5}**.

Combining Relations

We already know that **functions** are just **special cases** of **relations** (namely those that map each element in the domain onto exactly one element in the codomain).

If we formally convert two functions into relations, that is, write them down as sets of ordered pairs, the composite of these relations will be exactly the same as the composite of the functions (as defined earlier).

Combining Relations

Definition: Let R be a relation on the set A. The powers R^n, n = 1, 2, 3, …, are defined inductively by

$R^1 = R$

$R^{n+1} = R^n \circ R$

In other words:

$R^n = R \circ R \circ \ldots \circ R$ (n times the letter R)

Combining Relations

Theorem: The relation R on a set A is transitive if and only if $R^n \subseteq R$ for all positive integers n.

Remember the definition of transitivity:

Definition: A relation R on a set A is called transitive if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ for $a, b, c \in A$.

The composite of R with itself contains exactly these pairs (a, c).

Therefore, for a transitive relation R, $R \circ R$ does not contain any pairs that are not in R, so $R \circ R \subseteq R$.

Since $R \circ R$ does not introduce any pairs that are not already in R, it must also be true that $(R \circ R) \circ R \subseteq R$, and so on, so that $R^n \subseteq R$.

n-ary Relations

In order to study an interesting application of relations, namely **databases**, we first need to generalize the concept of binary relations to **n-ary relations**.

Definition: Let $A_1, A_2, \ldots, A_n$ be sets. An **n-ary relation** on these sets is a subset of $A_1 \times A_2 \times \ldots \times A_n$.
The sets $A_1, A_2, \ldots, A_n$ are called the **domains** of the relation, and n is called its **degree**.