PROBABILITY & DISTRIBUTIONS

Worked Solutions

Mean, Variance & SD of a Discrete R.V.

Probability distribution of X:

X	4	5	6	8
P(X)	0.1	0.3	0.4	0.2

$E(X) = \Sigma X \cdot P(X)$

$= 4(0.1)+5(0.3)+6(0.4)+8(0.2) = 5.9$

$E(X^2) = \Sigma X^2 \cdot P(X)$

$= 16(0.1)+25(0.3)+36(0.4)+64(0.2) = 36.3$

$Var(X) = E(X^2) - [E(X)]^2 = 36.3 - 34.81 = 1.49$

5.9
Mean E(X)

1.49
Variance

1.221
SD

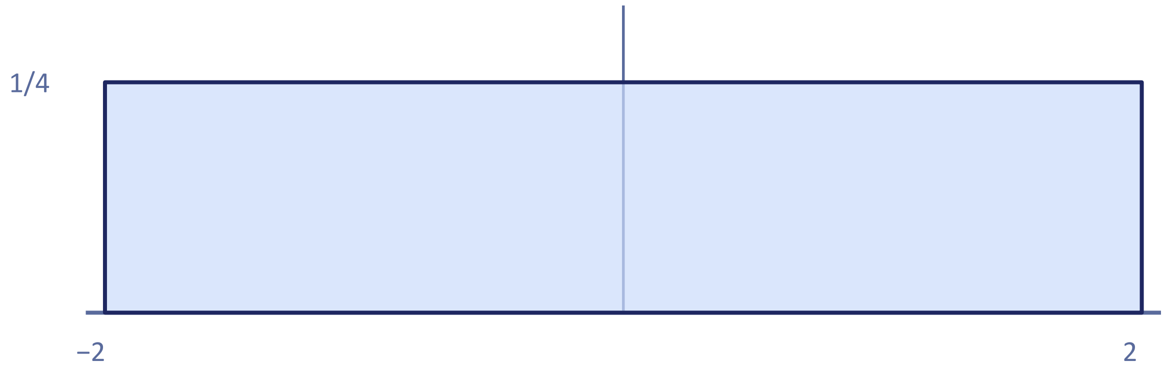
A random variable X has the following probability distribution:

X: 4, 5, 6, 8
P(X): 0.1, 0.3, 0.4, 0.2

Find the mean, variance and standard deviation of X.

Uniform Distribution on [−2, 2]

$f(x) = 1/4\,,\ -2 \le x \le 2$



3/4
P(X < 1)

3/4
P(|X−1|≥½)

i) P(X < 1)

$= \int_{-2}^{1} (1/4)\,dx = (1-(-2))/4 = 3/4$

ii) P(|X−1| ≥ 1/2) → X ≥ 3/2 or X ≤ 1/2

$P(X \ge 3/2) = 1/8,\quad P(X \le 1/2) = 5/8$

Total = 1/8 + 5/8 = 3/4

If the random variable X has the p.d.f.
$f(x)=1/4,\ -2 \le x \le 2$
$\quad 0,\ \text{otherwise}$

Find:
(i) $P(X < 1)$
(ii) $P(|X-1| \ge 1/2)$

QUESTION 6 · LONG ANSWER

Variance of a Discrete Distribution

X	8	12	16	20	24
P(X)	1/8	1/6	3/8	1/4	1/12

16
Mean

20
Variance

4.47
SD

$E(X) = \Sigma X \cdot P(X)$

$= 8(\frac{1}{8})+12(\frac{1}{6})+16(\frac{3}{8})+20(\frac{1}{4})+24(1/12)$

$= 1+2+6+5+2 = 16$

$Var(X) = E(X^2) - [E(X)]^2 = 276 - 256 = 20$

$SD = \sqrt{20} = 2\sqrt{5} \approx 4.47$

$E(X^2) = \Sigma X^2 \cdot P(X)$

$= 64(\frac{1}{8})+144(\frac{1}{6})+256(\frac{3}{8})+400(\frac{1}{4})+576(1/12)$

Find the variance and standard deviation for the following discrete distribution:

X=x: 8, 12, 16, 20, 24
P(X=x): 1/8, 1/6, 3/8, 1/4, 1/12

Laplace Distribution — Normalizing Constant & Variance

$f(x) = y_0\, e^{(-|x|)}$, $-\infty < x < \infty$

i) Finding y_0

$\int_{-\infty}^{\infty} y_0 e^{(-|x|)}\, dx = 2y_0 \int_0^{\infty} e^{(-x)} dx = 2y_0 = 1$

$\Longrightarrow y_0 = 1/2$

ii) Mean and Variance

Mean: $x \cdot e^{(-|x|)}$ is odd over $(-\infty,\infty) \Longrightarrow E(X) = 0$

Variance = $E(X^2) = \int_0^{\infty} x^2 e^{(-x)}\, dx = \Gamma(3) = 2! = 2$

$y_0 = ½$
Constant

$E(X)=0$
Mean

$Var=2$
Variance

The p.d.f. of a continuous random variable is
$f(x)= y_0\, e^{(-|x|)}$, $-\infty < x < \infty$

Find:
(i) the value of y_0
(ii) mean and variance of the distribution.

Standard Lognormal Distribution

$$f(x) = 1/(x\sqrt{(2\pi)}) \cdot e^{(-(\log x)^2/2)},\ x > 0 \ \ | \ \ f(x) = 0,\ x \le 0$$

This is the standard lognormal distribution: $\log X \sim N(0, 1)$, i.e. $\mu = 0$, $\sigma^2 = 1$.

Standard results for lognormal (μ, σ^2):

Mean: $E(X) = e^{(\mu + \sigma^2/2)}$

Variance: $Var(X) = (e^{\sigma^2} - 1)\cdot e^{(2\mu + \sigma^2)}$

With $\mu = 0$, $\sigma^2 = 1$:

$E(X) = e^{0.5} = \sqrt{e} \approx 1.649$

$Var(X) = (e-1)e = e^2 - e \approx 4.671$

$\sqrt{e} \approx 1.649$ Mean	≈ 2.161 Std. Deviation
≈ 4.671 Variance	*$SD = \sqrt{(e^2 - e)}$*

A continuous distribution is given by

$f(x)= (1/(x\sqrt{(2\pi)}))\ e^{(-(\log x)^2 /2)},\ x>0$
$\quad 0,\ x<0$

Find the mean and standard deviation of the distribution.

QUESTION 9 · LONG ANSWER

Validity & CDF of $f(x) = |x|$ on $[-1, 1]$

Step 1 — Show f(x) is a valid p.d.f.

- $f(x) = |x| \geq 0$ for all x ✓
- $\int_{-1}^{1} |x|\, dx = 2\int_0^1 x\, dx = 2[x^2/2]_0^1 = 1$ ✓

Hence f(x) is a valid probability density function.

Step 2 — Distribution function F(x)

For $-1 \leq x < 0$: $F(x) = \int_{-1}^{x}(-t)dt = (1-x^2)/2$

For $0 \leq x < 1$: $F(x) = \frac{1}{2} + \int_0^x t\, dt = (1+x^2)/2$

F(x) =

0	$x < -1$
$(1 - x^2)/2$	$-1 \leq x < 0$
$(1 + x^2)/2$	$0 \leq x < 1$
1	$x \geq 1$

Show that the function
f(x)=|x| for x in [-1,1] and zero elsewhere
is a possible p.d.f. and find the corresponding distribution function.

QUESTION 10 · LONG ANSWER

Expectations in an Alternating Die Game

A and B throw an ordinary die alternately (A first); the first to throw a 6 wins Rs. 121.

P(A wins) — A wins on throw 1, 3, 5, … :

$P(A) = 1/6 + (5/6)^2(1/6) + (5/6)^4(1/6) + \cdots$

$= (1/6) / (1 - 25/36) = (1/6)/(11/36) = 6/11$

P(B wins) = 1 − 6/11 = 5/11

Expectations (stake = Rs. 121):

$E(A) = 121 \times 6/11 =$ Rs. 66

$E(B) = 121 \times 5/11 =$ Rs. 55

6/11	5/11
P(A wins)	P(B wins)
Rs. 66	**Rs. 55**
E(A)	E(B)

Two persons throw an ordinary die alternately and the first who throws 6 is to get Rs. 121. Find their expectations.